Mathematical Reviews

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Mathematical Reviews

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HISTORY

★von Erhardt-Siebold, Erika and von Erhardt, Rudolf. The Astronomy of Johannes Scotus Erigena. Williams and Wilkins, Baltimore, Md., 1940. 69 pp.

★von Erhardt-Siebold, Erika and von Erhardt, Rudolf. Cosmology in the "Annotationes in Marcianum." More Light on Erigena's Astronomy. Williams and Wilkins, Baltimore, Md., 1940. 45 pp.

The first of these closely related studies attempts to show that the belief in Erigena's anticipation of essential ideas of Tycho Brahe's planetary system cannot be upheld. A passage in Erigena's "De Divisione Naturae" [9th century A.D.], usually interpreted as heliocentric, is shown to be contradictory to other parts of Erigena's writings. It is made evident that the passage in question permits a nonheliocentric interpretation, mainly by showing that "circa" can be translated in a very undetermined sense as "in the neighborhood." In addition to this result, many passages from ancient and medieval literature are discussed in order to explain details and background of Erigena's astronomical doctrine.

The second book discusses a commentary on Martianus Capella which was probably written by Erigena. This commentary was published recently by Cora E. Lutz [The Mediaeval Academy of America, Cambridge, Mass., 1939]. There, two passages occur, stating that the planetary orbits have their centers in the sun. The authors undertake to find an interpretation of these passages which does not contradict their above-mentioned conviction of the nonheliocentric character of Erigena's astronomy.

O. Neugebauer (Providence, R. I.).

von Erhardt, Rudolf and Erika. Archimedes' Sand-Reckoner. Isis 34, 214–215 (1943). [MF 8274]

The subtitle reads: "The following lines reply to Dr. O. Neugebauer's criticism in Isis, vol. 34, 4–7, 1942, of our article in vol. **33**, 578–602, 1942." Cf. these Rev. **3**, 257 and 258.

Miller, G. A. A fifth lesson in the history of mathematics. Nat. Math. Mag. 17, 212–220 (1943). [MF 8021]

de Losada y Puga, Cristóbal. Galileo. Rev. Univ. Católica Perú 10, 253–282 (1942). (Spanish) [MF 8056]

Kienle, H. Das Weltsystem des Kopernikus und das Weltbild unserer Zeit. Naturwissenschaften 31, 1–12 (1943). [MF 8175]

Moorman, R. H. The influence of mathematics on the philosophy of Descartes. Nat. Math. Mag. 17, 296–307 (1943). [MF 8247]

Ivins, William M., Jr. Two first editions of Desargues. Bull. Metrop. Mus. Art N.S. 1, 33–35 (1942). (10 plates) [MF 8481]

Ivins, William M., Jr. A note on Girard Desargues. Scripta Math. 9, 33–48 (1943). [MF 8344]

Report on Desargues' work. Photographs are given of the figures in the recently discovered complete copy of the Stereotomy which is now in the Metropolitan Museum in New York. [Cf. the preceding title.] *O. Neugebauer.*

Chant, C. A. Isaac Newton: Born three hundred years ago. J. Roy. Astr. Soc. Canada 37, 1–16 (1943). [MF 8002]

Dehn, Max and Hellinger, E. D. Certain mathematical achievements of James Gregory. Amer. Math. Monthly 50, 149–163 (1943). [MF 8161]

"In this paper we shall discuss Gregory's expansions of general and particular functions into series. In addition, we shall exhibit the ideas which are set forth in his first mathematical publication "Vera circuli et hyperbolae quadratura." These ideas are concerned, to some extent, with the associated problem of constructing by certain limiting processes the functions which measure the areas of circles and conics."

From the authors' summary.

Rufus, W. Carl. David Rittenhouse as a mathematical disciple of Newton. Scripta Math. 8, 228–231 (1941). [MF 8064]

Birkhoff, George D. Sir Joseph Larmor and modern mathematical physics. Science (N.S.) 97, 77–79 (1943). [MF 7948]

Karpinski, Louis C. Supplement to the bibliography of mathematical works printed in America through 1850. Scripta Math. 8, 233–236 (1941). [MF 8065]

Vera, Francisco. Chronological tables for Spain in the 16th century. Archeion 24, 403–437 (1942). (Spanish) [MF 8081]

Chronologically arranged list of dates of events and publications.

Whitman, E. A. Some historical notes on the cycloid. Amer. Math. Monthly 50, 309–315 (1943). [MF 8332]

Obituary: Oskar Bolza. Science (N.S.) 97, 108–109 (1943). [MF 7949]

Obituary: S. A. Chaplygin. Vestnik Akad. Nauk SSSR **1942**, 86–90 (1942). (Russian) [MF 8347]

Obnorski, S. P. Boris Mikhailovich Liapounoff. Vestnik Akad. Nauk SSSR **1942**, 83–85 (1942). (Russian) [MF 8346]

"Article on the occasion of the 80th anniversary of his birthday."

FOUNDATIONS

Martin, R. M. A homogeneous system for formal logic. J. Symbolic Logic 8, 1–23 (1943).

The author describes a new system of logic called the system H, which is designed to serve as a foundation for mathematics. It differs from the Whitehead-Russell system in not requiring a theory of types, and from systems like that of Zermelo in not having a universal undefined membership relation. The two principal categories of H are terms and formulas. The formulas are statements about the terms. The system is called homogeneous because all terms are of the same type. This single type, which the author calls the type of individuals, may also be thought of as the type of all classes whose members are individuals. Among the primitive notions are inclusion between terms, denoted by $x \supset y$ (this is a formula), negation and disjunction of formulas and universal quantification of a formula over the category of all terms. Two further primitive notions of an unusual sort are introduced to overcome the handicap of the restriction of terms to a single type. One, denoted by $x1F(x)$, is interpreted to mean the term which is the logical sum of all unit terms such that the formula $F(x)$ holds. The other, symbolized by $(xy; zw)zRw$, is a formula signifying that x is in the relation of the ancestral of the relation R to y.

The notions of class and of membership are not primitive but are defined contextually as in Whitehead and Russell. The notation $a\epsilon x$ for membership is significant only when x is an abstract such as $y\ni F(y)$, meaning the class of all terms y such that $F(y)$ holds. By definition $a\epsilon(y\ni F(y))$ means $F(a)$. This treatment of the membership relation serves the purpose of the theory of types since it prevents the construction of classes and relations, such as those leading to the paradoxes, which would violate the restrictions of type. On the basis of appropriate rules and axioms for this framework, the author proceeds to the derivation of the basic theorems of Boolean algebra and arithmetic. The development of arithmetic, which includes a theory of descriptions for the inductive cardinals and a theory of recursive relations for multiplication and exponentiation, is facilitated by the fact that a notation can be introduced for the quantification of arithmetic formulas over the class of all finite cardinals. The corresponding development of the arithmetic of the rational and real number systems on this basis is postponed to another paper. *O. Frink.*

Newman, M. H. A. Stratified systems of logic. Proc. Cambridge Philos. Soc. 39, 69–83 (1943). [MF 8144]

Direct criteria are presented in this paper for the "correct typing" of a general class of formal logics, including (with slight modifications) the system of Principia Mathematica, the simple theory of types in the form due to Church and several systems due to Quine. The discussion is restricted wholly to the "typing" of existing systems and is in no way concerned with the further possible relaxation of type or stratification restrictions.

By a type Newman understands an expression (of a system) whose constituent symbols are (say) Greek letters. The only principle for constructing further types is to enclose a row of Greek letters in parentheses. A primitive type consists of a single letter. If a symbol has no type, its type is said to be null. If α is $(\alpha_1\cdots\alpha_k)$, the α_i are factors of α. An r-fold type is (by recursion) either a primitive type or a type with r factors, each of which is an r-fold type. Any equation $X_0=\Phi X_1\cdots X_k$ determines certain positional relations among the letters X_m and X_n, denoted by $X_m\gamma_i X_n$. Given any equation $X=\Phi X_1\cdots X_k$, there will exist at most a finite number of the γ_i, say r of them, and each will satisfy the condition that, if $X_m\gamma_i X_n$ follows from the equation, then $X_m\gamma_h X_{n_h}$ does also, for $h=1, 2, \cdots, r$ and for some X_{n_h} $(1\leqq n_h\leqq k)$. Let the type of X be denoted by $\tau(X)$. If $\tau(X)$ is neither null nor primitive, $\tau_i(X)$ will denote the ith factor of the type of X. The general conditions for a "correct typing" are then: (1) each letter has the same type (or none) at each of its occurrences; (2) $\tau(X)$ is either null or an r-fold type (where r is the number of the γ_i-relations); (3) if $X\gamma_i Y$, then $\tau(X)$ is $\tau_i(Y)$; (4) a letter satisfying no γ_i-relation has the null type. Thus, for example, for the system of Quine's "New Foundations," a single primitive type is sufficient and further types may be obtained by enclosing it in any number of parentheses. The type of Z in an equation of the form $X=(Y\epsilon Z)$ is one higher than that of Y by (3).

Some of the theorems Newman proves are to the effect, roughly, that any equation $X=\Phi X_1\cdots X_k$ determines the type of X in terms of the types of $X_1, \cdots, X_k$; if $\tau(X)$ and $\tau(Y)$ are identical in a correct typing of a set of equations E, and E' results from E by substituting X everywhere for Y, then the same typing of E' is correct; a necessary and sufficient condition that a set of equations admit a correct typing is that it be stratified; if τ^1 and τ^2 are correct typings of any two sets E_1 and E_2 and $\tau^1(X)$ is $\tau^2(X)$ (for all common letters), then the combined typing is correct in $E_1\cup E_2$. Several more special theorems are also proved, applicable primarily to the formulation of the simple theory of types due to Church. *R. M. Martin* (Princeton, N. J.).

Curry, Haskell B. Some advances in the combinatory theory of quantification. Proc. Nat. Acad. Sci. U. S. A. 28, 564–569 (1942). [MF 7648]

This paper reports in somewhat greater detail the present state of some researches which the author briefly summarized in his presidential address [J. Symbolic Logic 7, 49–64 (1942); these Rev. 3, 289]. These researches deal with the problem of introducing into a primitive combinatory logic (or the equivalent λ-calculus) undefined terms denoting generality, in such a way that consistency is still provable. Paradoxes which might otherwise arise are avoided by restricting by rule the application of the new terms to a class of so-called canonical terms.

Three alternative methods of introducing such notions of generality are considered. The first involves the new term F denoting functionality, the second the term Ξ denoting formal implication and the third the terms Π and P denoting universal quantification and implication, respectively. For each of the resulting systems rules and postulates for the new terms and rules for canonical terms are given. For the last two systems the postulates which the author gives are considered to be merely tentative, since they do not seem to imply a deduction theorem of the desired strength. For these last two systems a consistency proof similar to that of Gentzen is sketched; the first system admits of a simpler proof like that of Church and Rosser. Finally, for the second system the possibility is discussed of introducing the notion of "canonicalness" by means of a new term H instead of by rule. *O. Frink.*

Alban, M. J. Independence of the primitive symbols of Lewis's calculi of propositions. J. Symbolic Logic 8, 25–26 (1943).

It is proved in this paper that the three primitive symbols "·", "$\sim$" and "$\Diamond$" of each of the Lewis systems of strict

implication S1–S6 are independent. (S6 is the system arising from B1–B8 and $\Diamond\Diamond p$.) That "·" is not definable in terms of "∼" and "◇" is immediate. For the proof for S1–S5 that "∼" is not definable in terms of "·" and "◇," the author constructs a special case of a matrix due to Henle, and modifies it slightly for the proof for S6. That "◇" is not definable in terms of "∼" and "·" follows from the result of Dugundji that none of the Lewis systems possesses a finite characteristic matrix. *R. M. Martin.*

Turing, A. M. The use of dots as brackets in Church's system. J. Symbolic Logic 7, 146–156 (1942).

The author generalizes H. B. Curry's conventions on the use of dots as brackets [J. Symbolic Logic 2, 26–28 (1937)], develops the general theory of this extension and applies it to Church's formulation of the simple theory of types [J. Symbolic Logic 5, 56–68 (1940); these Rev. 1, 321]. The principal departure from Curry's rules is to allow a group of a smaller number of dots to have seniority over a larger group, provided it is adjacent to an operator of higher power. This proves to be convenient in systems like that of Church. The paper also treats in greater detail than did Curry's the case of an operator denoted by simple juxtaposition and the rules for supplying omitted dots. It is shown by examples that greater clarity can sometimes be achieved by using more dots than are necessary, or by not replacing all the brackets by dots. *O. Frink.*

Dotterer, Ray H. A supplementary note on the rules of the antilogism. J. Symbolic Logic 8, 24 (1943).

The author modifies his previous set of rules for testing generalized antilogisms [J. Symbolic Logic 6, 90–95 (1941); these Rev. 3, 131] so that they are necessary as well as sufficient. *O. Frink* (State College, Pa.).

Quine, W. V. On existence conditions for elements and classes. J. Symbolic Logic 7, 157–159 (1942).

If the elementhood principle *200 is dropped from the system of the author's Mathematical Logic [W. W. Norton and Co., New York, 1940; these Rev. 2, 65], the resulting framework is known to be consistent. Quine compares this framework with the systems of Bernays and Zermelo. The connection with the Bernays system and the related von Neumann system is rather obvious, since all three involve the distinction between element and nonelement. However, to make the comparison it is necessary to identify the Bernays "sets" with their corresponding "classes," and even then the Bernays axioms of class existence are weaker than Quine's. The connection with the Zermelo system is not so apparent, but is actually closer. Quine shows that his own framework is equivalent to what the Zermelo framework would be with just two axioms of class existence: the Aussonderungsaxiom and an axiom guaranteeing the existence of a most inclusive class (Zermelo actually did not assume the latter). From this point on the two theories diverge; Quine's requires further axioms of elementhood, and Zermelo's, which has no concept of elementhood, requires further special axioms of class existence.
O. Frink (State College, Pa.).

Bernays, Paul. A system of axiomatic set theory. J. Symbolic Logic 7, 133–145 (1942).

This is the fourth installment of the author's system of set theory, which is essentially a simplification of the von Neumann system [for the previous installments, see J. Symbolic Logic 2, 65–77 (1937); 6, 1–17 (1941); 7, 65–89 (1942); these Rev. 2, 210; 3, 290]. His aim is to develop general set theory as far as possible without the use of the sum axiom (Vc), the power axiom (Vd) and the axiom of infinity (VI), and in such a way that the basic concepts are independent of the notion of cardinal number. He is careful to point out which proofs do not require the axiom of choice (IV).

The elementary part of set theory, dealing with the arithmetic operations of sum, product and power of classes and sets, is presented first on the basis of the pair class axiom, which states that the class $A\times B$ is represented by a set if A and B are represented by sets. Next a proof of the Schröder-Bernstein theorem is given. This leads naturally to the related subjects of cardinal number, comparability and the well-ordering theorem, which of course require the axiom of choice. On the basis of the axioms used here it is proved that any two sets are comparable, but to prove that any two classes are comparable requires stronger axioms. A cardinal number is defined to be an ordinal number for which there exists no lower ordinal of equal power. Two proofs of the well-ordering theorem are given, based on Zermelo's first and second proofs. Proofs are also given of the generalized Julius König theorem of Zermelo, and of a result called the theorem of adapted numeration. In connection with the Cantor theorem concerning the set of all subsets of a set, it is made clear how the set-theoretic paradoxes are avoided by means of the distinction between classes and sets which is characteristic of the author's system. *O. Frink* (State College, Pa.).

Blaquier, Juan. The axiom of Zermelo. Anales Acad. Nac. Ci. Ex. Fíz. Nat. Buenos Aires 8, 23 pp. (1942). (Spanish) [MF 8156]

ALGEBRA

Stabler, E. R. Boolean algebra as an introduction to postulational methods. Amer. Math. Monthly 50, 106–110 (1943). [MF 8053]

The author illustrates the pedagogical soundness of introducing postulational methods to the student by the study of Boolean rings, especially through comparison of ring-theoretic and lattice-theoretic sets of postulates.
G. Birkhoff (Cambridge, Mass.).

Fisher, R. A. Some combinatorial theorems and enumerations connected with the numbers of diagonal types of a Latin square. Ann. Eugenics 11, 395–401 (1942). [MF 7974]

Eighty-six years ago, Cayley tabulated the number of trees having s nodes (and therefore $s-1$ branches) when one of the nodes is specialized (as a "root" for the tree) [Philos. Mag. (4) 13, 172–176 (1857)]. He showed that this number a_s is given by equating successive coefficients in the formula

$$a_1+a_2x+a_3x^2+\cdots=(1-x)^{-a_1}(1-x^2)^{-a_2}(1-x^3)^{-a_3}\cdots.$$

Taking logarithms of both sides, the present author ex-

presses this in the form

$$\log\{\varphi(x)/x\}=\varphi(x)+\tfrac{1}{2}\varphi(x^2)+\tfrac{1}{3}\varphi(x^3)+\cdots,$$

where $\varphi(x)=a_1x+a_2x^2+\cdots$. He extends Cayley's table ($a_1=a_2=1$, $a_3=2$, $a_4=4$, $\cdots$, $a_{13}=12486$) by finding $a_{14}=32973$, $a_{15}=87811$, $a_{16}=235381$. By an ingenious elaboration of the same method, he finds also the number of "rings of branches of weight s," which is a_s plus the number of connected graphs with s nodes and s branches; for example, for $s=4$ it is $4+5=9$. [He uses the word "branch" in a different sense from that of Cayley, who meant merely one of the lines or arcs of the graph.] Calling this number b_s, he shows that the number of possible diagonal types for an $n\times n$ Latin square can be obtained as the coefficient of x^{n-1} in the expansion of

$$(1-x)^{-b_1}(1-x^2)^{-b_2}(1-x^3)^{-b_3}\cdots;$$

for example, for $n=17$ it is 3799624. *H. S. M. Coxeter.*

Fisher, R. A. Completely orthogonal 9 × 9 squares. A correction. Ann. Eugenics **11**, 402–403 (1942). [MF 7975]

The orthogonal 9×9 squares given in the author's book [Design of Experiments, Oliver and Boyd, Edinburgh, 1935] and in Fisher and Yates' "Statistical Tables for Biological, Agricultural and Medical Research" [Oliver and Boyd, Edinburgh, 1938] were erroneously stated to be of the same type [Ann. Eugenics **11**, 290–299 (1942); these Rev. **4**, 27]. Bose and Nair [Sankhyā **5**, 361–382 (1941); these Rev. **4**, 33] found that the corresponding finite projective geometries are Desarguesian and non-Desarguesian, respectively.

H. S. M. Coxeter (Toronto, Ont.).

Mann, Henry B. The construction of orthogonal Latin squares. Ann. Math. Statistics **13**, 418–423 (1942). [MF 7871]

It is well known that the regular representation of a group of order m can be regarded as an $m\times m$ Latin square [Cayley, Messenger of Math. **19**, 136 (1890)]. The author shows how certain mappings of the group into itself (not necessarily automorphisms) determine a set of orthogonal Latin squares. He remarks that nearly all such sets of squares so far constructed are based on Abelian groups of order p^n and type $(1, 1, \cdots, 1)$, using automorphisms. The new method is illustrated by using two groups of order 12 to construct new 12×12 Eulerian squares. One of these groups is the direct product of the four-group and the group of order 3; the other is the tetrahedral group. [The proof of theorem 4 seems to make the unwarranted assumption that T is an inner automorphism.] *H. S. M. Coxeter.*

Montel, Paul. Sur le nombre des combinaisons avec répétitions limitées. C. R. Acad. Sci. Paris **214**, 139–141 (1942). [MF 7839]

Let there be n objects $E_1, E_2, \cdots, E_n$. The author studies the number of orderings of the objects in groups of p, the object E_i appearing at most a_i times ($a_1+a_2+\cdots+a_n=p$). Various identities are given. The question has previously been studied by P. Sergescu [Acad. Roum. Bull. Sect. Sci. **23**, 485–491 (1941); these Rev. **3**, 259]. *W. Feller.*

Oldenburger, Rufus. Expansions of quadratic forms. Bull. Amer. Math. Soc. **49**, 136–141 (1943). [MF 7996]

The author has defined the characteristic of a quadratic form Q to be the maximum number σ of linearly independent forms L_i such that the rank of $Q+\lambda_1L_1^2+\cdots+\lambda_\sigma L_\sigma^2$ is the same as the rank of Q for all values of the λ's. He now proves that, if Q is of rank r, the smallest value of τ for which $Q=\sum_1^\tau L_iM_i$, where L_i and M_i are linear forms, is $r-\sigma$. The form Q is of characteristic σ if and only if $Q=G+H$, where G has characteristic σ and rank 2σ, while H has characteristic 0 and rank $r-2\sigma$. That is, Q has the Witt decomposition

$$Q=\sum_{i=1}^{\sigma} y_iz_i+\sum_{i=1}^{r-2\sigma} \nu_iu_i^2.$$

C. C. MacDuffee (New York, N. Y.).

Hodge, W. V. D. Some enumerative results in the theory of forms. Proc. Cambridge Philos. Soc. **39**, 22–30 (1943). [MF 7965]

Let $\|x_i^j\|$, $i=0, 1, \cdots; r$, $j=0, 1, \cdots, k$, be a $r+1$ by $k+1$ matrix whose elements are indeterminates. By a k-connex of type $(l_0, l_1, \cdots, l_k)$ is meant a homogeneous polynomial in the x_i^j, which is of degree l_j in the variable of the set $(x_0^j, x_1^j, \cdots, x_{r+1}^j)$ and which is at the same time a homogeneous polynomial in variables x_i^0 of the first set and in the determinantal forms $|x_i^0x_j^1|$, $|x_i^0x_j^1x_k^2|$, $\cdots$, involving the first two, or the first three, etc., sets of variables $(i<j<k<\cdots)$. These are the k-connexes which have been considered by the author in a previous paper [same Proc. **38**, 129–143 (1942); these Rev. **3**, 305]. For geometric applications the following generalization is introduced. It is assumed that of the elements $x_0^j, x_1^j, \cdots, x_r^j$ the first α_{k-j} elements are zero, while the remaining elements are indeterminates. Here $\alpha_0, \alpha_1, \cdots, \alpha_k$ are fixed integers satisfying the inequalities $\alpha_0>\alpha_1>\cdots>\alpha_k=0$ and $k-i\leqq\alpha_i\leqq r-i$. Then k-connexes are defined as above. The special k-connexes which are power products in the above determinantal forms naturally form a basis for all k-connexes of a given type, but they are not linearly independent. If the factors of such a power product are so arranged that the determinantal forms of order k come first, then the determinants of order $k-1$, etc., then a standard power product is one in which, for a suitable arrangement of the factors, the indices of equally placed columns in the various determinantal factors form nondecreasing sequences. The main result is to the effect that the standard power products of a given type $(l_0, l_1, \cdots, l_k)$ are linearly independent and form a base for all k-connexes of that type. From this the number of linearly independent k-connexes is derived. The connection with geometry arises from the fact that, in the special case $l_0=l_1=\cdots=l_k=n$, a k-connex is simply a form in the determinants of order $k+1$ of degree n, and hence represents a primal of order n in the ambient space S_N of the Grassmannian of the k-spaces of $[r]$. As to the above generalization of the idea of k-connexes, it is associated with the consideration of loci on the Grassmannian which correspond to a Schubert condition [cf. J. London Math. Soc. **17**, 48–64 (1942); these Rev. **4**, 52]. *O. Zariski.*

Schwerdtfeger, H. On contact transformations associated with the symplectic group. J. Proc. Roy. Soc. New South Wales **76**, 177–181 (1942). [MF 8183]

An unsuccessful attempt to prove a well-known theorem algebraically. If $y=Ax+Bp$, $q=Cx+Dp$, A, B, C, D n-rowed matrices, is a symplectic transformation, then its determinant is $+1$. The usual proofs involve continuity considerations. The algebraic proof of this paper, for the cases where either A or B is nonsingular, is correct. The

proof for the general case is based on a lemma which is not valid. It claims that, if the $n\times 2n$ matrix (A, B) is of rank n, there exists a diagonal matrix $S=(s_1, s_2, \cdots, s_n)$ such that $AS+B$ is of rank n. The expansion of $|AS+B|$ consists of 2^n terms in $s_1, s_2, \cdots, s_n$ whose coefficients are n-rowed determinants of (A, B), while there are $C_n^{2n}>2^n$ (if $n>1$) such determinants in (A, B). Hence $|AS+B|$ may vanish identically without one or more determinants of (A, B) being zero. *M. S. Knebelman* (Pullman, Wash.).

Taussky, Olga and Todd, John. Infinite powers of matrices. J. London Math. Soc. 17, 146–151 (1942). [MF 8261]

Let S be an algebra of order n over the real field with basis e_i, $i=1, 2, \cdots, n$, and let $\Lambda(S)$ be the set of elements

$$x=(x_1, x_2, \cdots, x_n)=\sum_{i=1}^{n} x_i e_i$$

of S such that $x^m\to(0, 0, \cdots, 0)$ in the ordinary sense of convergence in R_n as m tends to ∞. By applications of the structure theorem on algebras necessary and sufficient conditions that $\Lambda(S)$ be (i) bounded, (ii) closed and (iii) convex are obtained. These conditions are for (i) that S be expressible, and for (ii) and (iii) that S be expressible modulo its radical as a direct sum of algebras each isomorphic to the real field R, the complex field C or the field of real quaternions. Finally, it is shown that the condition that S be expressible modulo its radical as the direct sum of algebras isomorphic to R or C is necessary and sufficient for the elements in a regular representation of S to have a property analogous to the property P defined by McCoy for matrices over an algebraically closed field [N. H. McCoy, Bull. Amer. Math. Soc. **45**, 740–744 (1939); these Rev. **1**, 37]. *J. Williamson* (Flushing, N. Y.).

Abstract Algebra

Richardson, A. R. The class-ring in multiplicative systems. Ann. of Math. (2) **44**, 21–39 (1943). [MF 8072]

The author considers general types of partitions of a multiplicative system into disjoint classes. As examples one may think of coset expansions or similarly classes in a group. Various properties of such partitions are derived. The case where the classes may be considered to form an algebra is studied particularly. A characteristic polynomial can be introduced for the classes and conditions are obtained for this polynomial to have linear factors. For the special purpose of studying the factorization of the characteristic polynomial the author introduces an interesting method of matrix factorization of homogeneous polynomials in several variables, which promises to have applications in other problems. A linear matrix in m variables $x_1, \cdots, x_m$ is a matrix whose terms are linear in all x_i. It is shown that any homogeneous polynomial in an arbitrary number of variables can be factored into commutative linear matrix factors. A number of examples are given. The simplest is

$$ax^2+bxy+cdy^2=\begin{vmatrix} ax+by, & cy \\ -dy, & x \end{vmatrix}\cdot\begin{vmatrix} x, & -cy \\ dy, & ax+by \end{vmatrix},$$

and the proof of the general theorem proceeds by induction from this identity. A number of applications to the characteristic polynomial are made, particularly for quaternion algebras and the symmetric group on 3 or 4 variables. *O. Ore* (New Haven, Conn.).

Levi, Howard. A characterization of polynomial rings by means of order relations. Amer. J. Math. **65**, 221–234 (1943). [MF 8199]

In a polynomial ring it is possible to introduce an ordering, based essentially on the notion of degree. In this paper the author makes a set of assumptions which are satisfied for a polynomial ring and proves, conversely, that a ring satisfying these assumptions is a polynomial ring. The author considers a commutative ring R, a fully ordered set M and a function μ mapping R on M. Concerning the order function μ he makes certain assumptions which are obviously satisfied by any polynomial ring. In addition he assumes that for every infinite sequence of elements $A_1, A_2, \cdots$ in R there exists an integer m and elements $C_1, \cdots, C_m$ such that $\mu(A_{m+1}+C_1A_1+\cdots+C_mA_m)<\mu(A_{m+1})$ and $\mu(C_iA_i)<\mu(A_{m+1})$. Under all these assumptions he proves that R is a polynomial ring over a coefficient ring which is a domain of integrity in which the Hilbert basis theorem holds. With somewhat stronger assumptions he proves that the coefficient ring is a field. *H. W. Brinkmann* (Swarthmore, Pa.).

Helmer, Olaf. The elementary divisor theorem for certain rings without chain condition. Bull. Amer. Math. Soc. **49**, 225–236 (1943). [MF 8217]

Let R be a Prüfer ring, that is, a domain of integrity in which every ideal with a finite basis is principal. For $a, b\epsilon R$ define $RP(a, b)$ to be a divisor a_1 of a such that $(a_1, b)=1$ while $(a_2, b)\neq 1$ for any nonunit divisor a_2 of a/a_1. If $RP(a, b)$ exists whenever $a\neq 0$, R is said to be an adequate ring. Domains with unique factorization are adequate, but the converse is not true; in particular, the chain condition need not hold. For adequate rings the author proves the elementary divisor theorem: for any matrix M there exist unimodular matrices X, Y such that XMY is in normal form. Whether the theorem holds in any Prüfer ring remains an open question, but the author does not give an example of a Prüfer ring which is not adequate. *I. Kaplansky.*

Mahler, K. On ideals in the Cayley-Dickson algebra. Proc. Roy. Irish Acad. Sect. A. **48**, 123–133 (1942). [MF 7883]

In this paper the author considers the nonassociative division algebra of order eight discovered by Cayley and studied in greater detail by Dickson [Ann. of Math. (2) **20**, 155–171 (1919); J. Math. Pures Appl. (9) **2**, 319–325 (1923)]. Employing Dickson's definition of integral elements in this algebra, the author shows that every left (or right) ideal in the ring of all integral elements is a principal ideal. Moreover [theorem 2], every left ideal is generated by $G_0=\delta G^*$, where δ is a rational integer and G^* is a suitably restricted integral element whose norm is either 1, 2 or 4. Very essential to the proof is the result [theorem 1] that any element X of the algebra may be so approximated by an integral element G that the norm $N(X-G)\leqq 15/16$. The author states that more detailed considerations give $\frac{1}{2}$ as the exact bound in the last inequality. *A. E. Ross.*

Thurston, H. S. The solution of p-adic equations. Amer. Math. Monthly **50**, 142–148 (1943). [MF 8160]

The author develops a method for solving an equation $f(x)=0$ with rational integral coefficients in a p-adic extension Ω_p of the rational field. A solution is of the form

$$\alpha=a_0+a_1p+a_2p^2+\cdots, \qquad 0\leqq a_i<p.$$

A sequence of polynomials $F_0(x)=f(x)$, $F_i(x)$ is constructed such that, if α is a solution of $f(x)=0$ in Ω_p, then

$$a_i+a_{i+1}p+a_{i+2}p^2+\cdots$$

is a solution of $F_i(x)=0$ in Ω_p, and consequently a_i is a solution of $F_i(x)\equiv 0 \bmod p$. As is the case with previous methods, it is quite possible thus to determine $a_0, a_1, \cdots, a_k$ only to find that $F_{k+1}(x)\equiv 0 \bmod p$ has no solution. But if $f(x)=0$ has only simple roots (that is, $f(x)$ and $f'(x)$ are relatively prime), it is possible to determine the number of solutions and to isolate them by the present method in a finite number of steps. *C. C. MacDuffee.*

Albert, A. A. Non-associative algebras. I. Fundamental concepts and isotopy. Ann. of Math. (2) 43, 685–707 (1942). [MF 7400]

While important results on special nonassociative algebras had been known for some time, there was nothing in existence that with some right could have been described as a general theory of nonassociative algebras. The author develops such a theory for the first time. One of the main difficulties is the large number of algebras; after all, every tensor of rank three with two cogredient and one contragredient indices defines an algebra. The new idea is to replace the concept of equivalence by a new concept, that of isotopy. For associative algebras with a unity quantity, the latter concept coincides with that of equivalence; in the general case algebras can be isotopic without being equivalent. The paper starts with a general discussion of the elementary property of algebras. An algebra $\mathfrak{A}$ of order n over a field $\mathfrak{F}$ is defined as an n-dimensional vector space $\mathfrak{L}$ together with n quantities γ_{ijk} in $\mathfrak{F}$; the product of two vectors $a=(\alpha_1, \cdots, \alpha_n)$ and $x=(\xi_1, \cdots, \xi_n)$ is defined as the vector $(\mu_1, \cdots, \mu_n)$ with $\mu_k=\sum\alpha_i\gamma_{ijk}\xi_j$, where i and j in the sum range over $1, 2, \cdots, n$. If a is an arbitrary element of $\mathfrak{A}$ and x is a fixed element, the correspondence $a\rightarrow a\cdot x$ is a linear transformation R_x on $\mathfrak{L}$, $a\cdot x=aR_x$. This transformation R_x is called a right multiplication; the set of all right multiplications is a linear subspace, the right multiplication space $R(\mathfrak{A})$ of the total matric algebra $(\mathfrak{F})_n$ of order n^2 over $\mathfrak{F}$. Similarly, left multiplications L_x are defined by $a\rightarrow x\cdot a$, so that $x\cdot a=aL_x$. The left multiplications L_x form the left multiplication space $L(\mathfrak{A})$ of $\mathfrak{A}$. The right and left multiplications together with the identity generate a subalgebra of $(\mathfrak{F})_n$ which is called the transformation algebra $T(\mathfrak{A})$.

On the basis of these concepts, subalgebras, ideals and divisors of zero are studied. An algebra $\mathfrak{A}$ is said to be simple if $\mathfrak{A}$ does not contain any ideal except 0 and $\mathfrak{A}$. As in the case of associative algebras, the zero algebra of order one is excluded (that is, the algebra of order one in which every product vanishes). An algebra $\mathfrak{A}$ is said to be central simple if for every extension field $\mathfrak{K}$ of $\mathfrak{F}$ the extended algebra $\mathfrak{A}_{\mathfrak{K}}$ is simple; here $\mathfrak{A}_{\mathfrak{K}}$ is the algebra of order n over $\mathfrak{K}$ which has the same constants of multiplication γ_{ijk} as $\mathfrak{A}$. An elegant proof of the following theorem is obtained. An algebra of order $n>1$ over $\mathfrak{F}$ is simple if and only if $T(\mathfrak{A})$ is the total matric algebra $(\mathfrak{C})_t$, where $\mathfrak{C}=\mathfrak{C}(\mathfrak{A})$ is a field of degree t over F and $n=st$. Moreover, $\mathfrak{A}$ is central simple over a subfield $\mathfrak{Z}$ of $(\mathfrak{F})_n$ if and only if $\mathfrak{Z}=\mathfrak{C}$. [In this connection, cf. N. Jacobson, Duke Math. J. 3, 544–548 (1937).] The author treats algebras with a left unity quantity and algebras with a unity quantity; the right (left) minimal polynomial of an element b of $\mathfrak{A}$ is introduced.

The second part of the paper deals with the new concept of isotopy. Let $\mathfrak{A}$ and $\mathfrak{A}_0$ be two algebras of the same order n; both may be regarded as consisting of vectors of the same space $\mathfrak{L}$. If $R_x^{(0)}$ is the right multiplication corresponding to the element x of $\mathfrak{A}_0$, while R_x as above is the right multiplication with x in $\mathfrak{A}$, then $\mathfrak{A}$ is said to be isotopic to $\mathfrak{A}_0$ if there exist three nonsingular linear transformations P, Q, C such that $R_x^{(0)}=PR_{xQ}C$. If L_x and $L_x^{(0)}$ are the left multiplications in $\mathfrak{A}$ and in $\mathfrak{A}_0$, then this condition can be replaced by $L_x^{(0)}=QL_{xP}C$. If C is the identity, the isotope $\mathfrak{A}_0$ of $\mathfrak{A}$ is called a principal isotope. Every isotope of $\mathfrak{A}$ is equivalent to a principal isotope. Of the numerous interesting results, the following may be mentioned. If $\mathfrak{A}$ and $\mathfrak{A}_0$ are principal isotopes, each having a unity element, then $T(\mathfrak{A})=T(\mathfrak{A}_0)$. If $\mathfrak{A}$ and $\mathfrak{A}_0$ are principal isotopes, each having a unity quantity, and if both $\mathfrak{A}$ and $\mathfrak{A}_0$ are regarded as consisting of the vectors of the same space $\mathfrak{L}$, then a linear subspace of $\mathfrak{L}$ is an ideal of $\mathfrak{A}$ if and only if it is an ideal of $\mathfrak{A}_0$. In particular, $\mathfrak{A}$ is simple if and only if $\mathfrak{A}_0$ is simple. If in the field $\mathfrak{F}$ irreducible polynomials of degree n exist, then every algebra of order n over $\mathfrak{F}$ with a unity quantity has a principal isotope $\mathfrak{A}_0$ which is simple and has neither left nor right ideals except $\mathfrak{A}_0$ and 0. The center of an algebra is the set of all elements z of $\mathfrak{A}$ such that (1) z commutes with every element of $\mathfrak{A}$, (2) the associative law holds for every product of three factors, one of which is z. Isotopic simple algebras with unity quantities have equivalent centers. An algebra $\mathfrak{A}$ with a unity quantity is associative if and only if every isotope with a unity quantity is associative and equivalent to $\mathfrak{A}$. However, there exist associative algebras without unity quantities which are isotopic but not equivalent. An algebra $\mathfrak{A}$ with a left unity quantity is associative if and only if $R(\mathfrak{A})$ is an algebra. If $\mathfrak{A}$ is an associative algebra of order $n>1$ with a unity element over an infinite field, then there exists a nonassociative isotope $\mathfrak{A}_0$ of $\mathfrak{A}$ with $R(\mathfrak{A}_0)=R(\mathfrak{A})$. This also shows the existence of nonassociative algebras $\mathfrak{A}_0$ for which $R(\mathfrak{A}_0)$ forms an algebra. Every division algebra is isotopic to a division algebra with a unity quantity (a division algebra being an algebra without divisors of zero). The concepts mentioned can be studied further in the special cases of commutative algebras, alternative algebras and Lie algebras.
R. Brauer (Toronto, Ont.).

Albert, A. A. Non-associative algebras. II. New simple algebras. Ann. of Math. (2) 43, 708–723 (1942). [MF 7401]

The main topic of this second paper [cf. the paper reviewed above] is a construction of a family of algebras which contain crossed products and Cayley algebras as a special case, and which yield many new algebras. Let $\mathfrak{A}$ be an algebra with a unity quantity e over the field $\mathfrak{F}$. An extending group $\mathfrak{G}$ for $\mathfrak{A}$ is any finite group of linear transformations G of $\mathfrak{A}$ (considered as a vector space) which leaves e invariant. A $\mathfrak{G}$-ideal $\mathfrak{B}$ of $\mathfrak{A}$ is an ideal of $\mathfrak{A}$ for which bG (with b in $\mathfrak{B}$, G in $\mathfrak{G}$) always lies in $\mathfrak{B}$ again. If $\mathfrak{A}$ has no $\mathfrak{G}$-ideals except 0 and $\mathfrak{A}$, then $\mathfrak{A}$ is said to be $\mathfrak{G}$-simple. The algebra $\mathfrak{A}$ is $\mathfrak{G}$-central if for every extension field $\mathfrak{K}$ of $\mathfrak{F}$ the extended algebra $\mathfrak{A}_{\mathfrak{K}}$ is simple with regard to the group $\mathfrak{G}_{\mathfrak{K}}$ induced in $\mathfrak{A}_{\mathfrak{K}}$ by the transformations of $\mathfrak{G}$. For every pair of elements S,T of $\mathfrak{G}$ choose an element $g_{S,T}$ of $\mathfrak{A}$ which is neither a right nor a left-divisor of 0.

The algebra $\mathfrak{C}$ to be constructed will consist of the elements $\alpha=\sum u_S\cdot a_S$, where S ranges over the elements of $\mathfrak{G}$, where the u_S are new symbols and where the a_S are arbitrary

elements of $\mathfrak{A}$. Two such elements α are equal if the corresponding coefficients a_S in both elements are equal for every S. Two such elements α are added by adding corresponding coefficients a_S. For the definition of multiplication it will be sufficient to define the products $(u_S\cdot a)(u_T\cdot x)$ for S,T in $\mathfrak{G}$, a and x in $\mathfrak{A}$. Let $\mathfrak{H}$ be a fixed subset of $\mathfrak{G}$ containing the identity and set

$$w(S, T, a, x)=\begin{cases} aT\cdot x, & \text{if } S \text{ in } \mathfrak{H},\\ x\cdot aT, & \text{if } S \text{ not in } \mathfrak{H}.\end{cases}$$

Set $v=w(S, T, a, x)$, $y=w(ST, I, g_{S,T}, v)$. Then the definition of multiplication is given by $(u_S\cdot a)(u_T\cdot x)=u_{ST}\cdot y$. The factor y lies in $\mathfrak{A}$; it is a product consisting of the three factors $g_{S,T}$, aT, x taken in an arrangement depending on which of the elements S and ST belong to $\mathfrak{H}$. In this manner the algebra $\mathfrak{A}$, the group $\mathfrak{G}$, the subset $\mathfrak{H}$ of $\mathfrak{G}$ and the extension set $\mathfrak{g}=\{g_{S,T}\}$ determine an algebra $\mathfrak{E}=(\mathfrak{A}, \mathfrak{G}, \mathfrak{H}, \mathfrak{g})$. If $\mathfrak{A}$ has order n and if $\mathfrak{G}$ has order m, then $\mathfrak{E}$ has order mn. This algebra $\mathfrak{E}$ becomes the ordinary crossed product if $\mathfrak{A}$ is a normal separable field over $\mathfrak{F}$, if $\mathfrak{G}$ is the Galois group, $\mathfrak{H}=\mathfrak{G}$, and if the $g_{S,T}$ satisfy the conditions for ordinary factor sets.

In the general case, the extension $\mathfrak{E}$ need not be simple even when $\mathfrak{A}$ is $\mathfrak{G}$-simple, nor need $\mathfrak{E}$ be central when $\mathfrak{A}$ is $\mathfrak{G}$-central. However, under suitable assumptions concerning $\mathfrak{A}$, $\mathfrak{H}$, $\mathfrak{G}$, theorems of that type can be proved. This requires the following definitions. A linear transformation S on the vector space $\mathfrak{A}$ is called an inner or an outer transformation for this algebra according as there is or is not a quantity $b\neq 0$ in $\mathfrak{A}$ such that $b\cdot x=xS\cdot b$. If this equation holds for b in the center of $\mathfrak{A}$ [cf. the preceding review], then S is called a semi-identity transformation for $\mathfrak{A}$. An algebra $\mathfrak{A}$ is semisimple if it has a unity quantity e, and if it is a direct sum of ideals which are simple algebras. If it is now assumed that (1) $\mathfrak{A}$ is a semisimple algebra which is $\mathfrak{G}$-simple, (2) I is the only semi-identity transformation in $\mathfrak{G}$ and (3) $\mathfrak{H}$ does not contain any inner transformation except I, then the author is able to prove that $\mathfrak{E}=(\mathfrak{A}, \mathfrak{G}, \mathfrak{H}, \mathfrak{g})$ is simple. If the assumptions are added that $\mathfrak{A}$ is separable (that is, that $\mathfrak{A}_{\mathfrak{R}}$ is semisimple for every extension field $\mathfrak{R}$ of $\mathfrak{F}$) and that $\mathfrak{A}$ is $\mathfrak{G}$-central, then $\mathfrak{E}$ is a central simple algebra [cf. the preceding review]. If the algebra $\mathfrak{A}$ itself is simple, then it is only necessary to assume that $\mathfrak{H}$ consists of the identity and outer transformations in order to show that $\mathfrak{E}$ is simple.

The proofs given yield new proofs, even in the associative case. Perhaps the most interesting of the new algebras are those where $\mathfrak{H}$ consists of I alone. The theorems mentioned show that $\mathfrak{E}$ is simple whenever $\mathfrak{A}$ is; $\mathfrak{E}$ is central simple whenever $\mathfrak{A}$ is. These latter algebras include the Cayley algebras. For the group $\mathfrak{G}$ any permutation group of degree n can be chosen. We have only to take care that the unity quantity e is the sum of the n basis elements. Then application of an element G of $\mathfrak{G}$ to $\mathfrak{A}$ is to mean the transformation induced by the permutation G of the basis elements of $\mathfrak{A}$. If $\mathfrak{A}$ is any algebra and $\mathfrak{G}$ an extending group for $\mathfrak{A}$, then $\mathfrak{G}$ induces a group of linear transformations of $\mathfrak{E}$ by setting $\sum u_S\cdot a_S\rightarrow\sum u_S\cdot(a_SG)$ (G in $\mathfrak{G}$). Hence we may construct new algebras starting from $\mathfrak{E}$ and using this group $\mathfrak{G}$ of transformations again. Thus an iterative process for the construction of a family of central simple algebras is obtained. In particular, this gives generalized crossed products $(\mathfrak{N}, \mathfrak{G}, I, \mathfrak{g}_1, \cdots, \mathfrak{g}_t)$, where $\mathfrak{N}$ is a normal separable field of degree r, $\mathfrak{G}$ is its automorphism group and the $\mathfrak{g}_i$ are all factor sets or merely any extension sets. This algebra has order r^t over $\mathfrak{F}$. Finally a list of unsolved problems is given.

R. Brauer (Toronto, Ont.).

Jacobson, N. Classes of restricted Lie algebras of characteristic p. II. Duke Math. J. **10**, 107–121 (1943). [MF 8105]

[The first part appeared in Amer. J. Math. **63**, 481–515 (1941); these Rev. **3**, 103.]

A large new class of normal simple Lie algebras L_i of characteristic p is defined. Each L_i consists of the "derivations" (or generalized infinitesimal automorphisms) of a commutative and associative linear algebra A_i of order $(p-1)^m$. A subclass of the A_i includes the group algebras of the elementary Abelian groups; the case where A_i is the group algebra of a cyclic group of order p was discovered by E. Witt. The variety of the L_i is shown in the author's theorem that, if $p>3$, then L_1 and L_2 cannot be isomorphic unless A_1 and A_2 are isomorphic. Numerous other results are obtained.

G. Birkhoff (Cambridge, Mass.).

Morozov, V. V. On a nilpotent element in a semi-simple Lie algebra. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 83–86 (1942). [MF 8008]

Let $\mathfrak{A}$ be a semi-simple Lie algebra over the field of all complex numbers. The author gives an inductive proof of the following theorem. If x_0 is a nilpotent element then there exists an element h in a subalgebra $\mathfrak{A}_1$ such that $[h\, x_0]=x_0$. The subalgebra is inductively defined using the decomposition of $\mathfrak{A}$ relative to the inner derivation determined by x_0. Moreover, an elegant proof is given for the fact that each nilpotent element is contained in a simple Lie algebra of order 3. As a consequence it is proved that A is representable in triangular form. [See also N. Jacobson, Ann. of Math. (2) **36**, 875–881 (1935).]

O. F. G. Schilling (Chicago, Ill.).

Morosov, W. W. On the centralizer of a semi-simple subalgebra of a semi-simple Lie algebra. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 259–261 (1942). [MF 8093]

The author considers the centralizers Z of the semisimple subalgebras of semisimple Lie algebras over the field of all complex numbers. He shows that any nilpotent element of Z is contained in a simple subalgebra of order three of Z. He then concludes that Z is the direct sum of a semisimple algebra and an Abelian algebra whose elements have only simple elementary divisors. *A. A. Albert* (Chicago, Ill.).

MacLane, S. and Schilling, O. F. G. Groups of algebras over an algebraic number field. Amer. J. Math. **65**, 299–308 (1943). [MF 8205]

Let F be a field of algebraic numbers, and let K/F be a normal extension of F which is represented as the join of two normal extensions K'/F, K''/F. Let $\mathfrak{S}$ be a class of algebras over F which is split by K; the authors discuss the possibility of representing $\mathfrak{S}$ in the form $S'\times S''$, where S' is split by K' and S'' by K''. They prove the unexpected result that the condition of possibility for being able to represent in this way every $\mathfrak{S}$ split by K may be formulated in terms of the Galois groups Γ, Γ', Γ'' of K/F, K'/F, K''/F; if T', T'' are the natural homomorphisms of Γ on Γ', Γ'', respectively, the condition is the following: if a primary number l^m is the order of an element in Γ' and of an element in Γ'', there exists $\sigma\epsilon\Gamma$ such that $T'\sigma$, $T''\sigma$ both have orders $\equiv 0(l^m)$. The authors then show that the previous condition

is a condition on the Sylow sub-groups of Γ', Γ'' and that it is always satisfied if these Sylow sub-groups are regular. It is proved by examples that the condition fails to be satisfied for some groups Γ and that it is not equivalent to the regularity of Γ (when Γ is an l-group!).

In the proof of theorem 1, it seems that the prime divisor q should be selected in such a way that $\rho(q)=0$ (mod 1); also the statement $s'\equiv s''$ (mod. 1) [p. 301, line 5 from the bottom] should be replaced by $s'+s''\equiv 0$ (mod 1).
C. Chevalley (Princeton, N. J.).

NUMBER THEORY

Kesava Menon, P. Some congruence theorems. Proc. Indian Acad. Sci., Sect. A. 16, 95–100 (1942). [MF 7567]

This paper is a continuation of two earlier papers of the author [J. Indian Math. Soc. (N.S.) 2, 332–333 (1937); Math. Student 8, 156–158 (1940); these Rev. 3, 65]. Using the same method as in the second of these papers, the author proves the following results.

Let p be an odd prime. Then

$$\sum_{v=1}^{p-1} \binom{v}{p} v^{\frac{p-1}{2}} \equiv \frac{p-1}{2}\{1-(p-1)!\} \pmod{p^2},$$

$$\sum_r r^{\frac{p-1}{2}} \equiv (-1)^{\frac{p+1}{2}} \frac{p-1}{2} \prod_r r \pmod{p^2},$$

$$\sum_n n^{\frac{p-1}{2}} \equiv (-1)^{\frac{p+1}{2}} \frac{p-1}{2} \prod_n n \pmod{p^2},$$

where r runs over the quadratic residues and n over the quadratic nonresidues mod p. Moreover

$$\{1!2!\cdots(p-1)!\}^2 \equiv \frac{(-1)^{\frac{p-1}{2}}}{p-1} \pmod{p^2}.$$

The paper contains some misprints; in particular, the formulation of theorem 10 is not correct. *A. Brauer.*

Gage, Walter H. An arithmetical identity for the form $ab-c^2$. Bull. Amer. Math. Soc. 48, 898–900 (1942). [MF 7507]

An identity of the Liouville type is obtained involving sums over the integral solutions a, b, c of the equation $ab-c^2=n$. Certain of these solutions are then related in the familiar manner to the class number $G(n)$ of the binary quadratic forms of determinant $-n$. Since, by setting $a=x+z$, $b=y+z$, $c=z$, $ab-c^2$ becomes $xy+yz+zx$, it is then possible to derive a formula for the number N of representations of n by the form $xy+yz+zx$ with positive integers x, y, z. The result is $N=3(G(n)-\frac{1}{2}\zeta(n))$, where $\zeta(n)$ is the number of divisors of n. The forms $xy+yz+2zx$ and $xy+2yz+2zx$ are handled in a similar manner.
H. W. Brinkmann (Swarthmore, Pa.).

Shapiro, Harold. An arithmetic function arising from the ϕ function. Amer. Math. Monthly 50, 18–30 (1943). [MF 7943]

This paper is concerned with the iterated Euler totient function $\phi^1(m)=\phi(m)$, $\phi^2(m)=\phi(\phi(m))$, $\cdots$, $\phi^n(m)=\phi(\phi^{n-1}(m))$, where $\phi(m)$ is the number of integers not greater than m and relatively prime to m. The number m is said to belong to class n in case $\phi^n(m)=2$. This fact is expressed by writing $C(m)=n$, thus defining a numerical function C. Denoting $C(1)$ and $C(2)$ by 0, every positive integer belongs to a unique finite class of integers. The author studies this class and its general structure. It is shown that the nth class consists of numbers greater than 2^n and that the largest member is $2\cdot 3^n$. Expressed otherwise,

$$\frac{\log x/2}{\log 3} \leqq C(x) < \frac{\log x}{\log 2}.$$

This result is due to S. S. Pillai [Bull. Amer. Math. Soc. 35, 837–841 (1929)]. To examine the structure of a class, the author defines three sections: (I) $2^n<m<2^{n+1}$, (II) $2^{n+1}\leqq m\leqq 3^n$, (III) $3^n<m\leqq 2\cdot 3^n$. Members of (I) are all odd; those of (III) are even. A number of theorems are proved giving properties of integers belonging to a definite section of their classes. For example, if an integer m belongs to section (I) of its class any divisor of m also belongs to the first section of its own class. Some theorems have to do with the prime numbers of a class; for example, $p<(43/81)3^{C(p)}$ holds for all primes p not of the form $p\ 2\cdot 3^a+1$. All Mersenne primes 2^k-1, except those for $k=2, 3, 5, 17$, belong to section (II) of their classes. Page 21 contains a short table giving all the members of the nth class up to $n=5$ together with the smallest members for $n=6, 7, 8$. The author conjectures that the smallest member of each class is, in every case, a prime. The first ten such numbers are 3, 5, 11, 17, 41, 83, 137, 257, 641, 1097 and are indeed prime numbers. *D. H. Lehmer* (Berkeley, Calif.).

Gunderson, N. G. Some theorems on the Euler ϕ-function. Bull. Amer. Math. Soc. 49, 278–280 (1943). [MF 8224]

Generalizing the result of U. Scarpis [Period. Mat. 29, 138–139 (1913)] that $n\,|\,\varphi(p^n-1)$, the writer shows that, if $a>b$, and $m=$product of the distinct prime factors of n, then $(n^2/m)\,|\,\varphi(a^n-b^n)$; also $n\,|\,\varphi(a^n+b^n)$. The proof depends on some results of G. D. Birkhoff and Vandiver [Ann. of Math. (2) 5, 173–180 (1904)]. [Note that at the bottom of p. 278 the quantity $A=a^p+b^p$.] *L. Carlitz.*

Lehmer, D. H. Recurrence formulas for certain divisor functions. Bull. Amer. Math. Soc. 49, 150–156 (1943). [MF 7999]

This paper points out that numerical functions giving the excess of the number of divisors of n of one sort over the number of divisors of n of a second sort are sums over divisors of certain periodic Lucas functions. The following general formula is developed:

$$(1)\quad \sum_{m\geqq 0} D(n-m(m+1)2)\frac{\sin(2m+1)\theta}{\sin\theta}$$
$$=\begin{cases}(q/4)\csc^2\theta(-1)^{k+1}\left\{\dfrac{k\cos(2k-1)\theta}{\cos\theta}-\dfrac{\sin 2k\theta}{\sin 2\theta}\right\} & \text{if } n=k(k-1)/2,\\ 0 & \text{otherwise,}\end{cases}$$

where
$$D(n)=\textstyle\sum_{\delta|n}U_\delta q^\delta\ (n\geqq 1),\quad D(n)=0\ (n\leqq 0);$$
$U_n=(a^n-b^n)/(a-b)$ is the general Lucas function in which

a and b are roots of $x^2-Px+Q=0$, $Q\neq 0$; $q=Q^{-\frac{1}{2}}$. Formula (1) is obtained in an elementary way from the famous Jacobi identity.

Numerous special cases of (1) are listed. For example, when $q=1$, $\theta=\pi/4$ we have

$$E_4(n)-E_4(n-1)-E_4(n-3)+E_4(n-6)+E_4(n-10)-\cdots = \begin{cases}(-1)^{[(k-1)/2]}[k/2] & \text{if } n=k(k-1)/2,\\ 0 & \text{otherwise,}\end{cases}$$

a formula due to Glaisher. The special cases $q=1$, $\theta=\arcsin i$ and $q=1$, $\theta=\arcsin (i/2)$ are connected with the Pell equation $x^2-2y^2=1$ and the Fibonacci series, respectively.

R. D. James (Saskatoon, Sask.).

Linnik, U. V. A remark on the least quadratic non-residue. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 119–120 (1942). [MF 8006]

This note contains a proof of the following theorem. Let $\epsilon>0$ be any fixed number; let the primes p for which there are no quadratic nonresidues on the segment $[1, p^\epsilon]$ be called exceptional. Then for N sufficiently large, the number of exceptional primes on the segment $[N^\epsilon, N]$ does not exceed $320\pi(g+2)^g g!$, $g=[2\epsilon^{-2}+1]$. The proof uses a lemma due to Vinogradow and a theorem of the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 292–294 (1941); these Rev. **2**, 349]. *R. D. James* (Saskatoon, Sask.).

This paper is identical with the one in p. 35.

Behrend, F. A. On the frequency of primes. J. Proc. Roy. Soc. New South Wales **75**, 169–174 (1942). [MF 7982]

T. S. Broderick [J. London Math. Soc. **14**, 303–310 (1939); these Rev. **1**, 41] has proved a result on the distribution of primes by methods involving only elementary properties of integers. The author [J. London Math. Soc. **15**, 257–259 (1940); these Rev. **2**, 249] made improvements in the method of proof and arrived at the inequalities $\frac{1}{8}<\pi(n)(\lambda(n)/n)<12$, where $\pi(n)=\sum_{p\le n}1$, $\lambda(n)=\sum_{\nu=1}^{n}1/\nu$. In the present paper further refinements are made and the inequalities

$$\tfrac{1}{3}<\pi(n)(\lambda(n)/n)<3$$

result. The proof consists in establishing the inequalities

$$\mu(2^n)-\lambda(n)/n<\pi(n)(\lambda(n)/n)<2\mu(2^n)+10/\lambda(n),$$

where $\mu(n)=\sum_{\nu=1}^{n}(-1)^{\nu-1}/\nu$, from which (1) follows.

R. D. James (Saskatoon, Sask.).

Siegel, Carl Ludwig. Contributions to the theory of the Dirichlet L-series and the Epstein zeta-functions. Ann. of Math. (2) **44**, 143–172 (1943). [MF 8281]

In the first part of this paper the author improves, generalizes and simplifies previous results of Hardy and Littlewood, of Kusmin and of his own. The author proves an asymptotic formula for $L(s)$ which is a generalization of the recently discovered formula of Riemann for $\zeta(\frac{1}{2}+it)$. The formula is proved by using a representation of $L(s)$, due to Kusmin but proved here in a different way, as the sum of two integrals.

Using some of the results obtained in the proof of this asymptotic formula, the author then deals with the zeros of $L(s)$. Let $N(\Delta t)$ denote the number of different zeros of odd order of $L(\frac{1}{2}+i\tau)$ in the interval $t-\Delta t<\tau<t$. Let Δt be a function of t such that, as $t\to\infty$, $t^{\frac{1}{2}}\log t/\Delta t\to 0$. Then the author proves that

$$\liminf_{t\to\infty}\frac{N(\Delta t)}{\Delta t}\geqq\frac{m}{\pi e\phi(m)},$$

where m is the modulus of the characters of $L(s)$ and $\phi(m)$ is Euler's function. The author also obtains refined results for the number $B(\Delta t, \epsilon)$ of zeros of $L(s)$, $s=\sigma+i\tau$, in the rectangle $t-\Delta t<\tau<t$, $\frac{1}{2}-\epsilon<\sigma<\frac{1}{2}+\epsilon$, where Δt satisfies the same requirement as above, and $\epsilon=o(\log\log t/\log t)$. He proves

$$\liminf_{t\to\infty}\frac{B(\Delta t, \epsilon)}{t^\epsilon\Delta t}\geqq\frac{1}{4}\frac{m}{\pi e\phi(m)}.$$

In the second part of the paper similar problems are considered for certain Epstein zeta-functions. However, as the author indicates, the methods used are quite different. The Epstein zeta-functions considered are those associated with positive quadratic forms of variables. The matrix of the quadratic form is denoted by $\mathfrak{S}$ and the zeta-function by $\zeta(s;\mathfrak{S})$. If $\mathfrak{S}$ is unimodular then the line $\sigma=\frac{1}{4}k$ corresponds in certain respects to the critical line $\sigma=\frac{1}{2}$ for the ordinary zeta-function. Let $\mathfrak{E}_k$ denote the unit matrix of k rows and let $\mathfrak{S}$ belong to the genus of $\mathfrak{E}_k$. Then the author's results indicate that, for $k\geqq 12$, the zeros of $\zeta(s;\mathfrak{S})$, $s=\sigma+i\tau$, in the strip $2\leqq\sigma\leqq\frac{1}{2}k-2$, $0<\tau<t$ number $t\log 2/\pi+O(1)$. Moreover, all except at most a finite number of these lie on the line $\sigma=\frac{1}{4}k$. The cases $k=4$ and $k=8$ are completely known from certain identities and the remaining cases $3<k<12$ can be discussed but require some numerical computation. For the case $k=3$ the method gives no results.

N. Levinson (Cambridge, Mass.).

Mordell, L. J. On sums of three cubes. J. London Math. Soc. **17**, 139–144 (1942). [MF 8259]

The known rational solutions of

$$x^3+y^3+z^3=\lambda a^3$$

for $\lambda=1, 2$ are given by x, y and z as certain quartic polynomials in a parameter t with rational coefficients. This led the author to the following theorem whose proof is the object of the paper: The equation

$$x^3+y^3+z^3=n$$

has no solutions with x, y, z as quartic polynomials in a parameter t with rational coefficients unless $n=2a^3$ or $n=a^3$, where a is a rational number, and then the only solutions are those previously known. *B. W. Jones.*

Estermann, T. A new proof of a theorem of Minkowski. J. London Math. Soc. **17**, 158–161 (1942). [MF 8263]

The author gives a new arithmetical proof for the well-known theorem of Minkowski on linear forms.

P. Erdös (Philadelphia, Pa.).

Pall, Gordon. On the product of linear forms. Amer. Math. Monthly **50**, 173–175 (1943). [MF 8164]

A new proof is given for the case $n=2$ of Minkowski's theorem on linear forms. If $F_1(x_1, \cdots, x_n), \cdots, F_n(x_1, \cdots, x_n)$ are n nonhomogeneous linear forms of determinant $D\neq 0$, with real coefficients, there are integral values for the variables $x_1, \cdots, x_n$ such that $|F_1\cdots F_n|=2^{-n}D$. The proof shows first that without loss of generality we may restrict ourselves to the canonical forms $F_1=x+\rho y-\kappa$, $F_2=-\sigma x+y-\lambda$, where $0<\sigma\leqq\rho<1$, and then that integers x, y may be chosen to satisfy simultaneously

(A) $-\frac{1}{2}<x+\rho y-\kappa\leqq\frac{1}{2}$,
(B) $|(x+\rho y-\kappa)(-\sigma x+y-\lambda)|\leqq(1+\rho\sigma)/4$.

The latter part of the proof is an algorithm on the succession of solutions of (A) each of which will also satisfy (B) for a

certain range of values for λ and shows that these ranges for λ include all values. *M. Hall* (Washington, D. C.).

Buchstab, A. On an additive representation of integers. Rec. Math. [Mat. Sbornik] N.S. **10** (52), 87–91 (1942). (Russian. English summary) [MF 7772]

The author considers the problem of representing any even number $2N$ in the form $2N=p+N'$, where p is a prime, and where the prime factors of N' exceed $\psi(N)$, some increasing function of N, and proves that for all sufficiently large N such a representation is possible with $\psi(N)=(\log N)^\lambda$, for any λ, and the number of such representations exceeds $cN/(\log N \log\log N)$ for some $c>0$. The author also proves that there exist infinitely many primes p such that the prime factors of $p+2$ exceed $(\log p)^\lambda$, for any λ. In the proof use is made of Brun's method, together with estimates by Page and Siegel of the number of primes in certain arithmetical progressions. *D. H. Lehmer* (Berkeley, Calif.).

Gupta, Hansraj. On an asymptotic formula in partitions. Proc. Indian Acad. Sci., Sect. A. **16**, 101–102 (1942). [MF 7568]

This paper gives a short elementary proof of the asymptotic formula $p_k(n)\sim\binom{n-1}{k-1}/k!$ due to Erdös and Lehner [Duke Math. J. **8**, 335–345 (1941); these Rev. **3**, 69]. Here $p_k(n)$ is the number of partitions of n into exactly k summands. *R. D. James* (Saskatoon, Sask.).

Salem, R. and Spencer, D. C. The influence of gaps on density of integers. Duke Math. J. **9**, 855–872 (1942). [MF 7935]

Let $\{u_n\}$ be a sequence of increasing integers, $u_1=0$. Let $W(x)$ be a nonnegative nondecreasing function. The authors define a sequence to have the complete gap property with respect to W if in any closed interval $(a, a+l)$, $a\geqq 0$, $l\geqq l_0>0$, there exists an open interval of length not less than $W(l)$ which contains no point of the sequence. The authors prove the following theorems. (1) If $\int_1^\infty W(x)dx/x^2=\infty$, then any sequence having the complete gap property with respect to $W(x)$ has density 0. If $\int_1^\infty W(x)dx/x^2<\infty$ and $W(x)/x^2$ decreases, then there exists a sequence of positive density having complete gap property with respect to $W(x)$. (2) The authors investigate $W(x)=\theta x$. If $\theta\geqq\frac{1}{2}$, there is no sequence having complete gap property with respect to $W(x)$. If $\frac{1}{3}\leqq\theta<\frac{1}{2}$, the sequence has to have density $(\log m)/m$, and, if $0<\theta<\frac{1}{3}$, the sequence must have density $m^{-\alpha}$, where

$$\alpha=\log\left(\frac{1-2\theta}{1-3\theta}\right)\Big/\log 2\left(\frac{1-2\theta}{1-3\theta}\right).$$

All these theorems are best possible. *P. Erdös.*

Spencer, D. C. The lattice points of tetrahedra. J. Math. Phys. Mass. Inst. Tech. **21**, 189–197 (1942). [MF 7761]

This paper is a sequel to a previous paper concerned with the 2-dimensional case [Proc. Cambridge Philos. Soc. **35**, 527–547 (1939); these Rev. **1**, 203] of the right triangle. The tetrahedron considered is the s-dimensional one bounded by the hyperplanes

$$\begin{cases} x_i=1, & i=1,2,\cdots,s,\\ \eta=\omega_1x_1+\cdots+\omega_sx_s, & \end{cases}$$

where η is a positive parameter. The enumerative function of the lattice points of this tetrahedron under discussion is

$$N_r^{(s)}(\eta)=N_r^{(s)}(\eta\,|\,\omega_1,\omega_2,\cdots,\omega_s),$$

where the summation extends over those sets of integers m for which $\eta\geqq m_1\omega_1+m_2\omega_2+\cdots+m_s\omega_s$, or, in other words, over the lattice points $(m_1, m_2, \cdots, m_s)$ of the tetrahedron. Here r need not be an integer, but the most interesting and useful case is $r=0$. Let $\lambda_1, \lambda_2, \cdots$ be numbers of the form $\lambda_n=m_1\omega_1+m_2\omega_2+\cdots+m_s\omega_s$ and let a_n be the number of such representations. Then

$$P(z)=\prod_{k=1}^{s}(1-\exp(-\omega_kz))^{-1}=\sum_{n=1}^{\infty}a_n\exp(-\lambda_nz).$$

Using Perron's formula, we obtain the following contour integral representation of $N_r^{(s)}(\eta)$:

$$(1)\quad N_r^{(s)}(\eta)=(\Gamma(r+1)/2\pi i)\times\int_{c-i\infty}^{c+i\infty}P(z)\exp\{z(\eta-\sum\omega_k)-(1+r)\log z\}dz,$$

where $c>0$. The principal singularity of the integrand is the origin. Deforming the path so that it embraces the negative real axis without enclosing any of the other singularities of $P(z)$, we obtain

$$N_r^{(s)}(\eta)=(-1)^s\zeta_s(-r,\eta\,|\,\omega_1,\cdots,\omega_s)+T_r^{(s)}(\eta),$$

where the ζ-function is that of E. W. Barnes [Trans. Cambridge Philos. Soc. **19**, 374–425 (1904)] and the "error term" $T_r^{(s)}(\eta)$ is the contribution to the integral (1) arising from the nonzero singularities of $P(z)$. The paper discusses problems of estimating T, whose precise behavior for large η is obviously bound up in the arithmetical properties of the real parameters ω_k. Two ω's are said to belong to the same class if their ratio is rational. According as the number of classes among the given ω's is one or more, $T_r^{(s)}(\eta)$ is $O(\eta^{s-1})$ or $o(\eta^{s-1})$ as $\eta\to\infty$. In the important case $\omega_k=\log p_k$, where the p_k are distinct primes, there are s classes and we have the slightly better result

$$T_r^{(s)}(\eta)=o(\eta^{s-1}(\log\eta)^{-1-r}).$$

If one considers the ω's as the coordinates of a point in s-dimensional space we have the following result. For almost all points $(\omega_1, \omega_2, \cdots, \omega_s)$,

$$T_r^{(s)}(\eta)=\begin{cases} O(\{\log\eta\}^{s+\epsilon}) & \text{if } r=0,\\ O(1) & \text{if } r>0,\end{cases}$$

so that, on the whole, $T_0^{(s)}(\eta)$ is of very small order as compared with $\zeta_s(0,\eta)$, which is a polynomial in η of degree s. This polynomial then (except for its constant term) is asymptotically the best approximation to $N_0^{(s)}(\eta)$ [see D. H. Lehmer, Duke Math. J. **7**, 341–353 (1940); these Rev. **2**, 149]. In conclusion the author raises a number of questions of Diophantine approximation suggested by the above problem. One of these is as follows. For $k>1$, can $\theta_1, \theta_2, \cdots, \theta_k$ be found so that

$$\prod_{i=1}^{k}|\sin m\theta_i|\geqq Km^{-1}$$

holds for some constant K and for all integers $m>0$? If not, what function will best replace m^{-1} so that the inequality is true for at least one set of θ's? *D. H. Lehmer.*

Hua, Loo-keng. The lattice-points in a circle. Quart. J. Math., Oxford Ser. **13**, 18–29 (1942). [MF 7446]

Let the number $R(x)$ of integer pairs u, v such that $u^2+v^2\leqq x$ be $\pi x+O(x^\vartheta)$. Titchmarsh [Proc. London Math. Soc. (2) **38**, 96–115 (1934)] obtained $\vartheta\leqq 15/46$. Vinogradow [Bull. Acad. Sci. URSS (7) **1932**, 313–336] erred in proving $\vartheta<17/53+\epsilon$. Using some of Titchmarsh's results, and some intricate inequalities, Hua obtains $\vartheta\leqq 13/40+\epsilon$. *G. Pall* (Montreal, Que.).

Delsarte, Jean. Sur le gitter fuchsien. C. R. Acad. Sci. Paris 214, 147–149 (1942). [MF 7842]

Let $A(x)$ denote the number of points of a Fuchsian lattice on or within a geodesic circle of area πx on a surface S of constant negative curvature $1/a$. This function is expanded in a series of hypergeometric functions of the variable $-x/4a^2$, the first two parameters being roots of a quadratic equation $X^2=X+\lambda_n a^2$ and the third being unity. The quantity λ_n is a characteristic value and the coefficient in the series the square of the corresponding characteristic function which satisfies the partial differential equation and the boundary condition that the value is the same at any two congruent points. When $a\to 0$ the series reduces to the well-known formula of Hardy and Littlewood for the number of lattice points in a circle, the hypergeometric function becoming a Bessel function. The behavior of $A(x)$ when x becomes large is discussed briefly. *H. Bateman.*

Hille, Einar. Gelfond's solution of Hilbert's seventh problem. Amer. Math. Monthly 49, 654–661 (1942). [MF 7815]

The author gives a clear exposition of the following well-known theorem of Gelfond: α^β is transcendental if α and β are algebraic, β is irrational and $\alpha\neq 0$ or 1. *P. Erdös.*

THEORY OF GROUPS

Miller, G. A. Determination of the subgroups of small index. Proc. Nat. Acad. Sci. U. S. A. 29, 25–28 (1943). [MF 7678]

The author obtains several theorems on subgroups of small index in a given group. If a group G, of order g, contains $g/2-1$ subgroups of index 2, its order is of the form 2^m. The largest number of subgroups of index 2 in a group whose order is not a power of 2 is $g/3-1$. If a group G contains an invariant subgroup of odd prime index p and also p conjugate subgroups of index p, then the order of the group is divisible by p^2 and the invariant subgroup itself contains p conjugate subgroups of index p. If G contains an invariant subgroup of index p and a noninvariant subgroup of different prime index q, then p divides $q-1$. When a transitive unique permutation group of degree n, in which the subgroups composed of all the permutations which omit one letter are of degree $n-1$, contains other subgroups of index n which involve no invariant proper subgroup of the entire group, then this permutation group admits outer automorphisms. *D. C. Murdoch* (Saskatoon, Sask.).

Miller, G. A. Possible groups of automorphisms. Proc. Nat. Acad. Sci. U. S. A. 29, 49–52 (1943). [MF 8023]

After proving that no group G has a cyclic group of odd prime order as its group of automorphisms H, the author discusses the case where H is assumed to be a prime power group. In this latter case G must be the direct product of its Sylow subgroups and all of these except possibly one are of prime order. *G. de B. Robinson* (Ottawa, Ont.).

Miller, G. A. Groups containing a prime number of conjugate subgroups. Proc. Nat. Acad. Sci. U. S. A. 29, 104–107 (1943). [MF 8210]

An infinite series of groups of order p^m, such that every group of the series contains exactly p noninvariant subgroups which belong to a single conjugate set, can be constructed by extending the cyclic group of order $p^{m-1}(m>3)$ by an operator of order p which transforms the cyclic group into itself and gives rise to a commutator of order p. If $p=2$ then this series contains all groups having the desired property, while if p is odd there are other such infinite systems. *G. de B. Robinson* (Ottawa, Ont.).

Lewis, F. A. Generators of permutation groups simply isomorphic with $LF(2,p^n)$. Bull. Amer. Math. Soc. 48, 907 (1942). [MF 7509]

A representation of the abstract group simply isomorphic with the special linear homogeneous group $SLH(2, p^n)$ is given as a permutation group on the p^n+1 cosets of a certain subgroup K. The generators of the quotient group isomorphic with $LF(2, p^n)$ are given explicitly in terms of a certain "inversion" transposition T and a set of permutations which correspond to the generators of the additive group of the field $GF[p^n]$. *J. S. Frame* (Meadville, Pa.).

Frucht, Roberto. Coronas of groups and their subgroups, with an application to determinants. Revista Union Mat. Argentina 8, 42–69 (1942). (Spanish) [MF 7254]

Let P and H be permutation groups of degree r and s, respectively. Then the corona $P[H]$ is the group of permutations of the rs letters

$$\begin{array}{c} x_1^{(1)},\ x_2^{(1)},\ \cdots,\ x_s^{(1)},\\ x_1^{(2)},\ x_2^{(2)},\ \cdots,\ x_s^{(2)},\\ \cdots\cdots\cdots\cdots\cdots\cdots,\\ x_1^{(r)},\ x_2^{(r)},\ \cdots,\ x_s^{(r)} \end{array} \tag{1}$$

of the form (2) $p\cdot h_1^{(1)}\cdot h_2^{(2)}\cdot\ \cdots\ \cdot h_r^{(r)}$, where p is a permutation of P applied to the rows of (1) (leaving the arrangement of the letters of each row unchanged) and $h_p^{(q)}$ is a permutation of H applied to the letters of the qth row of (1). It is shown how the product of two elements of $P[H]$ can be written in the canonical form (2); subgroups of $P[H]$ are investigated which correspond to subgroups of P; for example, there is a subgroup isomorphic with the direct product of r groups isomorphic with H corresponding to the identity of P. Numerous examples are discussed. It is shown that the permutations on the n^2 elements of a determinant which leaves its value unchanged form a corona.
H. S. Wall (Evanston, Ill.).

Frucht, Roberto. Coronas of groups and their subgroups, with an application to determinants. Union Mat. Argentina, Publ. no. 24, 30 pp. (1942). (Spanish) [MF 8127]

This paper is identical with the paper reviewed above.

Singer, James. A pair of generators for the simple group $LF(3,3)$. Amer. Math. Monthly 49, 668–670 (1942). [MF 7817]

This paper contains a proof that the simple group $LF(3,3)$, of order 5616, can be generated by two elements, one of order 13 and one of order 8. Noting that $LF(3,3)$ is simply isomorphic to the collineation group of the finite projective geometry $PG(2,3)$, the author exhibits two collineations of orders 13 and 8, respectively, and shows that an arbitrary collineation can be expressed as a product of powers of these two. *D. C. Murdoch* (Saskatoon, Sask.).

Frame, J. S. Double coset matrices and group characters. Bull. Amer. Math. Soc. 49, 81–92 (1943). [MF 7987]

In this, the author's third paper dealing with the double cosets of a finite group [Proc. Nat. Acad. Sci. U. S. A. 26, 132–139 (1940); Bull. Amer. Math. Soc. 47, 458–467 (1941); these Rev. 1, 161; 2, 307], he generalizes a theorem proved in the second paper. Consider any representation R of a group G as a group of permutation matrices of degree n. Such a representation of G will break up into transitive representations R^t of degree n^t, where $n=\sum n^t$; let us assume that the subgroup of G associated with the representation R^t is H^t of order h^t. If γ_α is any element of G, then in the set of $h^s h^t$ elements of the double coset $H^s\gamma_\alpha H^t$ each element will appear $h_\alpha{}^{st}$ times, where $h_\alpha{}^{st}$ is the order of the cross-cut of $H_\alpha{}^s=\gamma_\alpha{}^{-1}H^s\gamma_\alpha$ and H^t. Writing $H_\alpha{}^{st}=H^s\gamma_\alpha H^t/h_\alpha{}^{st}$, we have

$$H_\alpha^{rs}H_{\beta'}^{st}/h^s=\sum_\eta c_{\alpha\beta\eta}^{rst}H_\eta^{rt},$$

where $H_{\beta'}^{st}$ is the double coset inverse to H_β^{st} and $c_{\alpha\beta\eta}^{rst}$ are positive integers. For $r=t$ and fixed η, these c's are the elements of a square matrix $M^s(H_\eta{}^{ts})$ of degree $\mu^{ts}=\mu^{st}$. Writing

$$K^{st}=\sum_\eta b_\eta^t M^s(H_\eta^{tt}),$$

the theorem of the paper is contained in the equation:

$$(n^t)^m|K^{st}|=n^s n^t A^{st}\prod(\chi_i)^{m_i},$$

$m_i=\mu_i{}^s\mu_i{}^t$, $m=\sum m_i=\mu^{st}$ and A^{st} is an integer. This theorem states that the determinant of the matrix n^tK^{st} splits into factors χ_i which are linear in the parameters $b_\eta{}^t$, that these factors χ_i are essentially the characters of the irreducible representation Γ_i of G and that the multiplicity m_i of the linear factor χ_i is $\mu_i{}^s\mu_i{}^t$, where $\mu_i{}^t$ is the multiplicity with which Γ_i appears in R^t. As an illustration, the theorem is applied to the simple group G of order 25920, choosing H^1 and H^2 to be subgroups of order 576 and 960, respectively. This choice of subgroups leads to the determination of the characters of the irreducible representation of G of degree 20.

Incidental to the proof of the above theorem the author shows that the sum of the number of symbols left fixed by the individual permutations of a subgroup of any permutation group is greater than or equal to the corresponding sum for the permutations of any coset of that subgroup.

G. de B. Robinson (Ottawa, Ont.).

Levi, F. W. Ordered groups. Proc. Indian Acad. Sci., Sect. A. 16, 256–263 (1942). [MF 8134]

Every Archimedean ordered group is isomorphic to a subgroup of the additive group of real numbers, and is therefore Abelian. [For proofs of this theorem, cf. O. Hölder, Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 53, 1–64 (1901), in particular, pp. 13, 14; R. Baer, J. Reine Angew. Math. 160, 208–226 (1929), in particular, §2; H. Cartan, Bull. Sci. Math. (2) 63, 201–205 (1939); G. Birkhoff, Ann. of Math. (2) 43, 298–331 (1942), in particular, §§13, 14; these Rev. 4, 3.] The present investigation is concerned with a generalization of this theorem. Put $|a|=a$ or a^{-1} in such a way that $1\leqq|a|$ for every element a in the ordered group G. Then a is said to be infinitely small with regard to the element b in G if $|a|^n<|b|$ for every positive integer n. The elements a and b are termed comparable if neither of them is infinitely small with regard to the other one. Denote by $E=E(a)$ the set of all the elements in G that are infinitely small with regard to the elements $a\neq 1$ and by $A=A(a)$ the set of all those elements in G that are either infinitely small with regard to a or comparable to a. Both E and A are intervals and subgroups of G; E is a normal subgroup of A, the quotient group A/E isomorphic to a subgroup of the additive group of real numbers. It is shown, furthermore, that a group with Abelian central quotient group admits of a group ordering if, and only if, 1 is its only element of finite order.

R. Baer.

Alexandroff, A. D. On groups with an invariant measure. C. R. (Doklady) Acad. Sci. URSS (N.S.) 34, 5–9 (1942). [MF 7452]

If G is a topological group and $\mu(E)$ is defined for open sets $E\subset G$, then μ is said to be a left invariant measure in case $0\leqq\mu(E)\leqq\infty$, $\mu(E)>0$ if E is not void, $\mu(E)<\infty$ for some $E\neq 0$, $\mu(E_1)\geqq\mu(E_2)$ if $E_1>E_2$, $\mu(xE)=\mu(E)$ for $x\epsilon G$; if for some neighborhood U of the unit and some open sets E_1, E_2 we have $E_1U\cap E_2U=0$, then $\mu(E_1+E_2)=\mu(E_1)+\mu(E_2)$. The group G is locally bounded in case there exists a neighborhood V of the unit such that for every neighborhood U of the unit there are elements $x_1,\cdots,x_k$ of G with $V\subset\sum x_iU$. The fundamental result is that G has a left invariant measure if and only if G is locally bounded. The measure is not necessarily completely additive but may always be chosen so that $\mu(E_1+E_2)\leqq\mu(E_1)+\mu(E_2)$ and $\mu(E)=\inf\mu(EU)$ taken over neighborhoods U of the unit. Furthermore a left invariant measure with these properties is unique (up to a constant factor). If the group is bounded in the sense that for every neighborhood U of the unit there are elements $x_1,\cdots,x_k$ with $G\subset\sum x_iU$ then there exists a measure which is left invariant, right invariant and inversely invariant ($\mu(E)=\mu(E^{-1})$). Also in order that a left invariant μ exist with $\mu(G)<\infty$ it is necessary and sufficient that G be bounded and this in turn is equivalent to the statement that G is a subgroup of a direct product of groups of unitary matrices. A 1-1 correspondence between invariant measures and invariant integrals is stated.

N. Dunford (New Haven, Conn.).

ANALYSIS

Polya, G. On the zeros of the derivatives of a function and its analytic character. Bull. Amer. Math. Soc. 49, 178–191 (1943). [MF 8154]

This address delivered before the American Mathematical Society reviews the highlights of the still developing theory connecting the behavior of the zeros of the derivatives of a function with its analytic character. The results fall into two groups, one dealing with analytic functions of a complex variable, the other with real functions of a real variable. A number of the elegant results stated in the address have not previously been published. Several interesting conjectures and unsolved problems are presented, particularly in connection with the first part.

E. S. Pondiczery.

Whitney, Hassler. Differentiability of the remainder term in Taylor's formula. Duke Math. J. 10, 153–158 (1943). [MF 8108]

Let f be of class C^{n+p}, $1\leqq n\leqq n+p$, in the neighborhood of the origin. Let $x^nf_n(x)/n!$ be the remainder after n terms in Taylor's formula for $f(x)$ in powers of x. Define

$f_n(0)=f^{(n)}(0)$. Then f_n is of class C^p, and of class C^{n+p} if $x\neq 0$. Furthermore, $x^k f_n^{(p+k)}(x)\to 0$ as $x\to 0$ if $k=1, \cdots, n$. Conversely, if f_n is a function which is of class C^{n+p} for $x\neq 0$, and such that the limit of $x^k f_n^{(p+k)}(x)$ exists as $x\to 0$ if $k=0, \cdots, n$, there exists a function f of class C^{n+p} having $x^n f_n(x)/n!$ as remainder after n terms in Taylor's formula. The values of $f^{(k)}(0)$, $k=0, \cdots, n-1$, may be specified arbitrarily. Generalizations to many dimensions are indicated. *A. E. Taylor* (Los Angeles, Calif.).

Whitney, Hassler. Differentiable even functions. Duke Math. J. 10, 159–160 (1943). [MF 8109]

If f is an even function of x defined in a neighborhood of $x=0$, there exists a function $g(u)$ such that $g(x^2)\equiv f(x)$. It is shown that, if f is analytic, of class C^∞, or of class C^{2s}, g may be made analytic, of class C^∞, or of class C^s, respectively, in the neighborhood of $u=0$. If f is odd, it may be written as $xg(x^2)$, and if f is analytic, of class C^∞, or of class C^{2s+1}, g may be made analytic, of class C^∞, or of class C^s, respectively. *A. E. Taylor* (Los Angeles, Calif.).

Whitney, Hassler. The general type of singularity of a set of $2n-1$ smooth functions of n variables. Duke Math. J. 10, 161–172 (1943). [MF 8110]

It was shown [Ann. of Math. (2) 37, 645–680 (1936)] by the author of this paper that an n-dimensional differentiable manifold can be embedded in Euclidean space of $2n$ dimensions. Further, by a slight modification of such a mapping, a regular mapping may be obtained (that is, one which preserves the independence of every set of n independent directions selected at an arbitrary point). The present paper studies mappings of a region of Euclidean n-space into Euclidean $2n-1$ space. Such a mapping will in general possess singular points, where the mapping is not regular in the above sense. The investigations show that, by slight alterations of the mapping function, the singular points may be made isolated and virtually all of the same kind. The typical singularity is one which corresponds to the cross cap, which occurs when a projective plane is mapped into 3-space. In terms of a suitable local coordinate system the mapping at a singular point is of the form $y_1=x_1^2$, $y_2=x_2$, $\cdots$, $y_n=x_n$, $y_{n+i-1}=x_1x_i$, $i=2, \cdots, n$. The class of the mappings is discussed with the aid of the two papers by the same author reviewed above. *A. E. Taylor.*

Shohat, J. A. and Bushkovitch, A. V. On some applications of the Tchebycheff inequality for definite integrals. J. Math. Phys. Mass. Inst. Tech. 21, 211–217 (1942). [MF 8169]

Let $p(x)\geqq 0$, $f_1(x)$ and $f_2(x)$ monotonic, then an inequality of Tchebycheff states that

$$\int_a^b p(x)dx\cdot\int_a^b p(x)f_1(x)f_2(x)dx \gtreqless \int_a^b p(x)f_1(x)dx\int_a^b p(x)f_2(x)dx,$$

where the upper sign holds if both $f_1(x)$ and $f_2(x)$ are increasing or decreasing, the lower if one is increasing, the other decreasing. The authors apply this inequality to obtain estimates of the remainder term of Taylor's expansion, and also to Simpson's rule. *P. Erdös.*

Calculus

✶James, Glenn and James, Robert C. **Mathematics Dictionary.** Revised Edition. Digest Press, Van Nuys, Calif., 1943. viii+273 pp.; appendix, 46 pp.

The first edition was reviewed in these Rev. 3, 293. While some errors of the first edition have been corrected, others have been carried over without correction. Thus, the erroneous formulation of Abel's theorem mentioned in the above cited review has remained unchanged. *W. Feller.*

Babini, José. On the application of finite differences to the successive derivation of composite functions. Revista Union Mat. Argentina 8, 160–164 (1942). (Spanish) [MF 8027]

Let u and v be functions of x, let D denote the operator d/dx, let Δ_1 be a difference operator applying to p (for example, $\Delta_1 a^p=a^{p+1}-a^p$) and Δ_2 a similar difference operator applying to q and let $f(u, v)$ be a function of u and v. The author derives the formula

$$D^n u^p v^q f(u, v)=u^p v^q \sum \frac{u^r v^s}{r!s!}\frac{\partial^{r+s} f}{\partial u^r \partial v^s}\Delta_1^r\Delta_2^s u^{-p}v^{-q}D^n u^p v^q,$$

where the summation runs over integral values of r and s such that $0\leqq r+s\leqq n$. The formula is applied to a variety of examples. *W. E. Milne* (Corvallis, Ore.).

Dresden, Arnold. The derivatives of composite functions. Amer. Math. Monthly 50, 9–12 (1943). [MF 7942]

Tola Pasquel, José. On an elementary way of presenting the theory of convexity, of points of inflection and of maxima and minima of real variables. Rev. Univ. Católica Perú 10, 132–135 (1942). (Spanish) [MF 8057]

Herrera, Félix Eduardo and Balanzat, Manuel. Extension of Dirichlet's function to the complex domain. Revista Union Mat. Argentina 8, 155–159 (1942). (Spanish) [MF 8026]

The expression

$$\lim_{m\to\infty}\lim_{n\to\infty}(\cos m!\pi x)^{2n},$$

which is one for rational, and zero for irrational, real values of x, becomes infinite for complex values of x. *E. Helly.*

Gutiérrez Novoa, Lina. Geometrical applications of certain logarithmic integrals. Revista Soc. Cubana Ci. Fís. Mat. 1, 42–47 (1942). (Spanish) [MF 8308]

The value of $\int dy/y=\int d\ln y$ extended along a closed curve in a plane is zero if the integral is properly defined at the points where $y=0$. This fact applied to a closed polygon yields the generalized theorem of Menelaus for this polygon. Similar integrals in polar coordinates are connected with Ceva's theorem. *E. Helly* (Chicago, Ill.).

Theory of Series

✶Hyslop, J. M. Infinite Series. Oliver and Boyd, Edinburgh; Interscience Publishers, New York, 1942. xi+120 pp. $1.75. 1942

This is a concise textbook giving the essentials of the classic theory of convergence of real sequences and series.

The brevity of the text precludes extensive motivation of the definitions and theorems, but the examples worked out in the text and the problems at the ends of the chapters provide illumination. The text begins with definitions of bounds and limits and is, on the whole, precise and accurate. A few unfortunate wordings and lapses are found. On page 2, the author writes "there is at least one value of x for which $f(x)>K-\epsilon$, where ϵ is any positive number," when he means that "to each $\epsilon>0$ corresponds at least one value of x for which $f(x)>K-\epsilon$." The definition of convergence [p. 108] of a double series is unsatisfactory. Unless one adds the requirement that each term of the double series appears in the union of the author's "groups," no series can be "convergent" unless the terms are all 0; and when this requirement is imposed, only absolutely convergent series are "convergent." The book is designed for students having completed a first course in calculus; such a student should find it to be an intelligible introduction to basic methods in modern analysis as well as to the classic theory of convergence of series. *R. P. Agnew* (Ithaca, N. Y.).

Silberstein, Ludwik. On some infinite sets of numbers. Philos. Mag. (7) **34**, 32–34 (1943). [MF 8016]

An infinite sequence of numbers is defined such that the nth term is the arithmetic mean of the two preceding terms. From the solution of this defining difference equation it follows immediately that the limit of the sequence is a weighted mean of the first two terms. A corresponding result is obtained if the nth term is the mean of the three preceding terms. It is stated that the property generalizes to higher numbers. *J. L. Barnes* (Princeton, N. J.).

Barnhart, C. A. Geometric examples of convergent series. Nat. Math. Mag. **17**, 159–162 (1943). [MF 7775]

Walsh, C. E. Inequalities for positive series. Edinburgh Math. Notes no. **32**, 30–32 (1941). [MF 7552]

Using an elementary device, the author arrives at a result similar to the inequalities proved by Copson and Elliott [J. London Math. Soc. **1**, 93–96 (1926); **2**, 9–12 (1927); **3**, 49–51 (1928); **4**, 21–23 (1929)]. *O. Szász.*

Mayer, A. E. A mean value theorem concerning Farey series. Quart. J. Math., Oxford Ser. **13**, 48–57 (1942). [MF 7449]

Mayer, A. E. On neighbours of higher degree in Farey series. Quart. J. Math., Oxford Ser. **13**, 185–192 (1942). [MF 7917]

(1) The author shows that the numerators and denominators of two subsequent reduced fractions a_0/b_0, a_1/b_1 of the Farey series F_n of order n are "similarly ordered," that is, $(a_1-a_0)(b_1-b_0)>0$. In addition he proves a similar theorem for the second and third neighbors (in the latter case $n=4$ is an exception). Based on these results he establishes various comparison theorems for subsequent fractions t_0, t_1, t_2 in F_n comparing t_1 with the means $M_r(t_0, t_2) = \{\frac{1}{2}(t_0^r+t_2^r)\}^{1/r}$ of t_0 and t_2. For instance, we have either $t_1\leqq M_{-1}$ or $t_1\geqq M_1$. In case $t_0\geqq\frac{1}{2}$ we have either $t_1\leqq M_{-1}$ or $t_1>M_2$.

(2) In the second paper a conjecture of Hardy is confirmed, according to which generally the kth neighbors of F_n are similarly ordered provided $n\geqq N(k)$. A direct discussion furnishes the minimum values $N(k)=1, 1, 5, 10, 13$ for $k=1, 2, 3, 4, 5$, respectively. *G. Szegö.*

Bickley, W. G. and Miller, J. C. Note on the reversion of a series. Philos. Mag. (7) **34**, 35–36 (1943). [MF 8017]

If the relation

$$y=x+a/x+b/x^3+c/x^5+\cdots$$

becomes, upon reversion,

$$x=y+A/y+B/y^3+C/y^5+\cdots,$$

Bickley gives formulae expressing $A, B, \cdots, G$ in terms of $a, b, \cdots, g$ and Miller, by introducing certain parameters, puts these in suitable form for calculation. Checks on the accuracy are provided. *W. E. Milne* (Corvallis, Ore.).

Kac, M. Note on the partial sums of the exponential series. Univ. Nac. Tucumán. Revista A. **3**, 151–153 (1942). [MF 8150]

The author proves that for $x>2$ and constant ω

$$(*)\qquad \left|e^{-x}\sum_{k<x+\omega x^{\frac{1}{2}}}\frac{x^k}{k!}-(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\omega}e^{-u^2/2}du\right|<\frac{34\log[x]}{[x]^{\frac{1}{2}}}.$$

This inequality is, as the author shows, an immediate result of Cramér's estimate for the remainder term in the central limit theorem when applied to the Poisson distribution. [It is interesting to remark that, for $x\to\infty$, the left hand term is $O(x^{-\frac{1}{2}})$; cf. Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, Springer, Berlin, 1925, vol. 1, p. 80.] *W. Feller* (Providence, R. I.).

Bosanquet, L. S. Note on the Bohr-Hardy theorem. J. London Math. Soc. **17**, 166–173 (1942). [MF 8266]

Let

$$\Delta^\alpha\epsilon_n=\sum_{\nu=n}^{\infty}\binom{\nu-n-\alpha-1}{\nu-n}\epsilon_\nu=\sum_{\nu=0}^{\infty}\binom{\alpha}{\nu}(-1)^\nu\epsilon_{n+\nu}$$

and

$$S_n^\alpha=\sum_{\nu=0}^{n}\binom{n-\nu+\alpha-1}{\nu-n}s_\nu=\sum_{\nu=0}^{n}\binom{\nu+\alpha-1}{\alpha-1}s_{n-\nu},$$

where $s_n=a_0+a_1+\cdots+a_n$. If α, $p\geqq0$, then a sequence ϵ_n is such that $\sum\epsilon_na_n$ is summable (C, α) whenever $S_n^\alpha=o(n^{\alpha+p})$ if and only if (i) $\epsilon_n=O(n^{-p})$ and (ii) $\sum n^{\alpha+p}|\Delta^{\alpha+1}\epsilon_n|<\infty$. When these conditions are satisfied, the series $\sum S_n^{\rho-1}\Delta^\rho\epsilon_n$ is summable $(C, \alpha-\rho)$ when $0\leqq\rho\leqq\alpha$. The same result holds if O and o are interchanged. The author gives a new proof of this theorem, combining results and methods of several writers. *R. P. Agnew* (Ithaca, N. Y.).

Iyengar, K. S. K. Notes on summability. I. An equivalence theorem in a general field of summability. J. Mysore Univ. Sect. B. **3**, 123–129 (1942). [MF 7799]

Let $\varphi_n>0$, $\sum1/\varphi_n=\infty$, and $\varphi_n\to\infty$ as $n\to\infty$. Let $\delta a_n=a_n-a_{n-1}$. Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be nonzero constants. A sequence $s_1, s_2, \cdots$ is called summable (φ, k) to $s[s_n\to s, (\varphi, k)]$ if the difference equation

$$(1)\qquad (\varphi_n\delta+\alpha_1)(\varphi_n\delta+\alpha_2)\cdots(\varphi_n\delta+\alpha_k)y_n=\alpha_1\alpha_2\cdots\alpha_ks_n$$

possesses at least one solution $y_1, y_2, \cdots$ for which $y_n\to s$ as $n\to\infty$. Let $L_n(p)=\prod_{r=j}^{n}(1+p\varphi_r^{-1})$, where j is taken so great that no factor is 0 or negative. If one solution of (1) is of order $O[L_n(p)]$ or $o[L_n(p)]$, then one solution of the difference equation

$$(2)\qquad (\varphi_n\delta+\beta_1)(\varphi_n\delta+\beta_2)\cdots(\varphi_n\delta+\beta_{k'})y_n=s_n$$

will also be of order $O[L_n(p)]$ or $o[L_n(p)]$, respectively, provided that $k'\geqq k$, $\Re(\beta_r)\neq0$ and $\Re(p+\beta_r)\neq0$; the additional condition $\Re(\beta_r)>0$ implies that all solutions of (2)

must be of order $O[L_n(p)]$ or $o[L_n(p)]$, respectively. If one solution of (1) converges to s, then one solution of (2) converges to $[(\pi\alpha_r)/(\pi\beta_r)]s$ if $k'\geqq k$ and $\beta_r\neq 0$, and all solutions of (2) converge to the same number if $\beta_r>0$. If $s_n\rightarrow s$, (φ, k), then $\varphi_n\delta s_n\rightarrow 0$, $(\varphi, k+1)$. If $s_n\rightarrow s$, (φ, k), and $\varphi_n\delta s_n\geqq -A$ (A a constant), then $s_n\rightarrow s$. These theorems generalize the equivalence, Abelian and Tauberian theorems for Cesàro and Hölder summability to which they reduce when $\varphi_n=n$.
R. P. Agnew (Ithaca, N. Y.).

Rogosinski, W. W. On Hausdorff's methods of summability. II. Proc. Cambridge Philos. Soc. **38**, 344–363 (1942). [MF 7802]

In the first paper [same Proc. **38**, 166–192 (1942); these Rev. **3**, 296] the author studied a general theory of strength for Hausdorff matrix transformations. In this second paper he extends this theory to the continuous analogue of the Hausdorff matrix transformations, that is, the linear integral transforms

$$t(x)=\int_0^1 s(xt)d\phi(t)=\int_0^x s(t)d\phi(t/x), \qquad x>0.$$

Restrictions on $\phi(t)$ are the same as for matrix transformations, while $s(x)$ may be any function which is Borel-measurable and bounded in every finite interval $(0, X)$. As was shown in the first paper, the strength of a Hausdorff integral transformation depends essentially on the distribution of the zeros of the corresponding Mellin transform $T(z)=\int_0^1 t^z d\phi(t)$ in the half-plane $\Re z\geqq 0$, with $z=\infty$ included as a boundary point. Most of the preliminary remarks and results of this paper have already been established by this reviewer [Amer. J. Math. **64**, 208–214 (1942); these Rev. **3**, 233; Ann. of Math. (2) **43**, 501–509 (1942); these Rev. **4**, 80], in particular, those pertaining to the most significant parallelisms between Hausdorff matrix and integral transforms, regularity of the integral transforms and the permutability of these transforms. Many of the author's main results are the analogues of those obtained in the first paper. Of particular interest is an extension of a theorem of Silverman [Ann. of Math. (2) **21**, 128–140 (1920)]. The extension provides essentially a test for the strength of a continuous Hausdorff method of summation relative to that of a suitably chosen Hölder method H_n. Implications and applications of this theorem are given. *H. L. Garabedian.*

Glass, T. F. and Leighton, Walter. On the convergence of a continued fraction. Bull. Amer. Math. Soc. **49**, 133–135 (1943). [MF 7995]

A new convergence criterion is given for the continued fraction with complex elements. *J. Shohat.*

Leighton, Walter and Thron, W. J. Continued fractions with complex elements. Duke Math. J. **9**, 763–772 (1942). [MF 7927]

The authors give new sufficient conditions and a new set of convergence regions for the continued fraction

$$1+\frac{a_1|}{|1}+\frac{a_2|}{|1}+\cdots.$$

They further study the convergence to an analytic function of the continued fraction

$$1+\frac{a_1z|}{|1}+\frac{a_2z|}{|1}+\cdots,$$

z complex, by applying the theory of normal families to its approximants. *J. Shohat* (Philadelphia, Pa.).

Leighton, Walter and Thron, W. J. On the convergence of continued fractions to meromorphic functions. Ann. of Math. (2) **44**, 80–89 (1943). [MF 8076]

The authors discuss the convergence of the continued fraction

$$\frac{1|}{|1}+\frac{z_2-1|}{|1}+\frac{z_2(z_3-1)|}{|1}+\cdots$$

through the series formed by its approximants (Euler) A_n/B_n, namely,

$$\sum_{n=1}^{\infty}(A_nB_n^{-1}-A_{n-1}B_{n-1}^{-1})$$
$$=1+\sum_{n=2}^{\infty}(-1)^{n-1}(1-z_2^{-1})(1-z_3^{-1})\cdots(1-z_n^{-1}).$$

The results are applied to the continued fractions

$$\frac{1|}{|1}+\frac{a_2|}{|1}+\frac{a_3|}{|1}+\cdots,\quad 1+\frac{a_1x|}{|1}+\frac{a_2x|}{|1}+\cdots,$$

where a_r, x complex. *J. Shohat* (Philadelphia, Pa.).

Polynomials, Polynomial Approximations

Albert, A. A. An inductive proof of Descartes' rule of signs. Amer. Math. Monthly **50**, 178–180 (1943). [MF 8165]

Tschebotaröw, N. On a particular type of transcendent equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 38–41 (1942). [MF 7595]

The paper is concerned with the distribution of roots of equations of the form $g(z)\cosh z+h(z)\sinh z=0$, where $g(z)$ and $h(z)$ are real polynomials. The methods are extensions of those of Hurwitz, who considered the analogous problem with the hyperbolic functions replaced by circular functions [Mitt. Math. Ges. Hamburg **2**, 25–31 (1890)]. The details of the new results are obscure because they are stated in special terminology defined in a previous paper in volume **33** (1941) of the same journal; this volume is at present unavailable to the reviewer. *P. W. Ketchum* (Urbana, Ill.).

Tschebotaröw, N. On R-integrable polynomials. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 63–65 (1942). [MF 7611]

A function is of type π if it is an entire function of the form

$$(\beta_r/r!)x^re^{-\gamma x^2+\delta x}\prod_{\nu=1}^{\infty}(1+\delta_\nu x)e^{-\delta_\nu x}$$

with $\beta_r\neq 0$, the δ_ν real, $\gamma\geqq 0$. Laguerre [Oeuvres, vol. 1, pp. 174–177] assumed, and Pólya [Rend. Circ. Mat. Palermo **36**, 279 (1913)] proved, that functions of type π are the most general functions which can be approximated by polynomials with real roots. A polynomial (1) $a_0+a_1x+\cdots+a_nx^n$ is said to be R-prolongable if there exists a function of type π such that (1) is a segment of its expansion. A polynomial (2) $a_0y^n/n!+a_1y^{n-1}/(n-1)!+\cdots+a_{n-1}y+a_n$ is said to be R-integrable if it is a mth derivative of a polynomial with real roots. The new definition of R-prolongable polynomials given above allows the author to prove this general theorem: a polynomial (1) is R-prolongable if and only if (2) is R-integrable. A less general theorem (with

another definition of R-prolongable) was given by the same author in 1936 [Bull. Soc. Phys. Math. Kazan (3) 8 (1936–1937)]. *S. Mandelbrojt* (Houston, Tex.).

Marden, Morris. The zeros of certain composite polynomials. Bull. Amer. Math. Soc. 49, 93–100 (1943). [MF 7988]

Let $\rho>0$, $0<\lambda<1$, $A_0(z)$ be a given polynomial of degree m with all its zeros in $|z|\leqq r$; let $A_n(z)$ be defined by the recursion

$$A_k(z)=(\beta_k-z)A'_{k-1}(z)+(\gamma_k-k)A_{k-1}(z),\quad k=1,2,\cdots,n.$$

If $|\beta_k|\leqq\lambda r$ and $\rho|\gamma_k-k-m|\geqq|\gamma_k-k|+m\lambda$, then all zeros of $A_n(z)$ are in $|z|\leqq r\max(1,\rho^n)$. If the condition on γ_k is modified to $0<\rho|\gamma_k-k-m|\leqq|\gamma_k-k|+m\lambda$, $\rho\geqq1$, then all zeros of $A_n(z)$ are in

$$|z|\leqq r\prod_{k=1}^{n}\frac{|\gamma_k-k|+m\lambda}{|\gamma_k-k-m|}.$$

Theorems on entire functions are also established.
G. Szegö (Stanford University, Calif.).

Gaspar, Fernando L. On a property of algebraic equations with real roots. Revista Union Mat. Argentina 8, 81–90 (1942). (Spanish) [MF 7481]

The following is proved here. Given a polynomial $f(x)$ of degree n and with n distinct zeros $x_1, x_2, \cdots, x_n$ and given the weights $y_1, y_2, \cdots, y_n$ and moments $\mu_k=y_1x_1^k+y_2x_2^k+\cdots+y_nx_n^k$. Then the polynomials

$$P_r(x)=\begin{vmatrix} 1 & x & x^2 & \cdots & x^r \\ \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_r \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{r+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_{r-1} & \mu_r & \mu_{r+1} & \cdots & \mu_{2r-1} \end{vmatrix},\quad r=0,1,2,\cdots,n,$$

are orthogonal in the set of points $x_1, x_2, \cdots, x_n$ with respect to the weights $y_1, y_2, \cdots, y_n$; that is,

$$\sum_{i=1}^{n} y_iP_j(x_i)P_k(x_i)=0\quad (j\neq k \text{ or } j=k\geqq n)$$
$$\neq 0\quad (j=k<n)$$

Furthermore, $P_n(x)=cf(x)$, from which it is clear that, corresponding to each root x_k is a set of constants l_j such that

$$x_k^r=\sum_{j=1}^{n} l_j\mu_{j+r-1},\qquad r=0,1,\cdots,n.$$

[Reviewer's note: Compare above results with G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., vol. 23, New York, 1939, pp. 24–26; these Rev. 1, 14, where they are essentially included in theorems of more general form.] *M. Marden* (Milwaukee, Wis.).

Vijayaraghavan, T. On a theorem of J. L. Walsh concerning the moduli of zeros of polynomials. Proc. Indian Acad. Sci., Sect. A. 16, 83–86 (1942). [MF 7194]

Ostrowski proved recently [Bull. Amer. Math. Soc. 47, 742–746 (1941); these Rev. 3, 110] the following theorem: If $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a given polynomial and α is such that for $z=\alpha$ no term of $P(z)$ exceeds the sum of the moduli of the other terms, then α is a root of an equation $\sum b_\nu z^\nu=0$ with $|b_\nu|=|a_\nu|$, $0\leqq\nu\leqq n$. The author's proof is based on an argument similar to that of Ostrowski avoiding the application of Rouché's theorem. *G. Szegö.*

Kac, M. On the average number of real roots of a random algebraic equation. Bull. Amer. Math. Soc. 49, 314–320 (1943). [MF 8230]

Let $f(x)=a_0+a_1x+\cdots+a_nx^n=0$ be an algebraic equation. Suppose that $a_i=\pm1$; then Littlewood and Offord proved that almost all these equations (that is, with the exception of $O(2^n)$ equations) have less than $25(\log n^2)$ real roots. The author considers the following similar problem. Suppose that the a_i have normal distribution with density $e^{-u^2}/\pi^{\frac{1}{2}}$; then the mean value of the number of real roots of $f(x)$ equals

$$(4/\pi)\int_0^1\frac{[1-n^2[x^2(1-x^2)/(1-x^{2n})]^2]^{\frac{1}{2}}}{1-x^2}dx\sim(2/\pi)\log n.$$

P. Erdös (Philadelphia, Pa.).

Clarkson, J. A. and Erdös, P. Approximation by polynomials. Duke Math. J. 10, 5–11 (1943). [MF 8096]

Let $\{n_i\}$ be a set of distinct positive integers, $\sum n_i^{-1}<\infty$. Let M denote the set of polynomials with real coefficients containing only powers x^{n_i}. According to a theorem of Müntz-Szász, the uniform limit of polynomials M in $0\leqq x\leqq1$ constitutes only a subclass of all functions continuous in $(0,1)$. The authors show that the characteristic property of this subclass is that its functions $f(x)$ are analytic in the unit circle $|x|<1$ involving only powers x^{n_i}. The power series of $f(x)$ is in general not convergent at $x=1$. In the case of lacunary series ($n_{i+1}/n_i\geqq c>1$) the power series is uniformly convergent in the closed interval 0, 1. Finally an extension of the Müntz-Szász theorem is proved according to which the divergence of $\sum n_i^{-1}$ is necessary and sufficient in order that any continuous function in $a\leqq x\leqq b$ $(0<a<b)$ can be uniformly approximated by polynomials involving only powers x^{n_i}. *G. Szegö* (Stanford University, Calif.).

Laden, H. N. Fundamental polynomials of Lagrange interpolation and coefficients of mechanical quadrature. Duke Math. J. 10, 145–151 (1943). [MF 8107]

Let $\varphi_n(x)=a_n(x-x_1)\cdots(x-x_n)$ represent the classical orthonormal polynomial (of Jacobi (J), Laguerre (L), Hermite (H)), to which correspond a Lagrange interpolation formula with its fundamental polynomials

$$l_k(x)=\frac{\varphi_n(x)}{(x-x_k)\varphi_n'(x_k)}$$

and a Gaussian mechanical quadrature formula with its coefficients H_k, $k=1, 2, \cdots, n$. Using a method due to Sonine [C. Winston, Ann. of Math. (2) 35, 658–677 (1934)], also relations between the $l_k(x)$ and the H_k, the author is enabled to compare H_k and H_j, also $|l_k(x)|$ and $|l_j(x)|$, for different j and k and various properly specified x, also the behaviour of $(1-x_j^2)^mH_j^{-1}(J)$, $x_j^mH_j^{-1}(L_1H)$, etc., when j varies. *J. Shohat* (Philadelphia, Pa.).

Webster, M. S. A convergence theorem for certain Lagrange interpolation polynomials. Bull. Amer. Math. Soc. 49, 114–119 (1943). [MF 7991]

Suggested by similar theorems of Rogosinski and G. Grünwald, the author proves the following. Let $f(x)$ be continuous in $-1\leqq x\leqq+1$ and let $L_n(f;x)=L_n(x)$ denote the corresponding Lagrange polynomial of degree $n-1$ coinciding with $f(x)$ at the points $x=\cos\theta_k$, $\theta_k=k\pi/(n+1)$, $k=1, 2, \cdots, n$. Then

$$\lim_{n\to\infty}\{L_n(\theta-\theta_1/2)+L_n(\theta+\theta_1/2)\}=2f(\cos\theta),\quad 0<\theta<\pi,$$

uniformly in every interval $\epsilon \leqq \theta \leqq \pi-\epsilon$, $0<\epsilon<\pi/2$. A counter-example is constructed showing that this is not valid for $\theta=0$ or $\theta=\pi$. *G. Szegö* (Stanford University, Calif.).

Shabde, N. G. On an integral involving Laguerre functions. Bull. Calcutta Math. Soc. **34**, 53–54 (1942). [MF 7528]

The integral

$$\int_0^1 L_n(xu)L_n(-xu)L_m[x(1-u)]L_m[x(u-1)]du$$

is a polynomial of degree $2(m+n)$ whose coefficients are expressed by means of simple series with the aid of theorems of Howell and Adams. *H. Bateman.*

Special Functions

✶Snow, Chester. The Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory. National Bureau of Standards, U. S. Department of Commerce, Washington, D. C., 1942. $2.00.

This lithographed volume contains an outline of the theory of the ordinary hypergeometric function with special reference to the associated Legendre functions. Numerous formulae pertaining to these functions are given. A special feature is the detailed discussion of the analytic continuation of the hypergeometric function to all parts of the complex plane, its three parameters remaining unrestricted. An extensive chapter is devoted to a treatment of the transformations and properties of the associated Legendre functions which are frequently required in mathematical physics. This material supplements, in an important manner, the discussion of these functions given in the standard treatises such as Hobson's, Bateman's, etc. Other topics discussed are: Heun's differential equation and its solutions; generalizations of Fourier's integral; certain integral equations of potential theory related to the associated Legendre functions; certain coördinate systems (e.g. toroidal, annular, etc.) and their inversion.

M. A. Basoco (Lincoln, Neb.).

Giraud, Georges. Sur les zéros de certaines fonctions de Bessel et de Whittaker. C. R. Acad. Sci. Paris **214**, 649–651 (1942). [MF 7888]

The author is concerned with solutions of the differential equation

$$\text{(1)} \qquad \frac{d^2u}{dz^2}+\left(\frac{1}{4}-\frac{h}{z}+\frac{m-m^2}{z^2}\right)=0, \qquad h, m \text{ real},$$

which by a change of variable may be seen to be related to certain confluent hypergeometric differential equations which have been studied by Whittaker [Whittaker, E. T. and Watson, G. N., A Course of Modern Analysis, Cambridge, England, 1920, chap. 16]. For $h\neq 0$, (1) has the solution

$$\text{(2)} \qquad \Gamma(2m)F_{h,m}(z)=\exp(-izm/2)M_{ih,m-\frac12}(z),$$

where $M_{\lambda,\mu}(z)$ is Whittaker's function. In order to avoid having $F_{0,m}(z)$ become identically zero for $m=0, -1, -2, -3, \cdots$, the author takes as the definition of $F_{0,m}(z)$ the following expression

$$\text{(3)} \qquad F_{0,m}(z)=2^{2m-1}z^{\frac12}J_{m-\frac12}(z/2),$$

where $J_\mu(z)$ is a Bessel function. He outlines a proof of the following theorem. "A necessary and sufficient condition for the equation $z^{-m}F_{0,m}(z)=0$ to have imaginary roots is that $2m<-1$ and not be equal to an odd integer. Moreover, if $|m+p|<\frac12$, p a positive integer, then this equation has precisely $2p$ imaginary roots. If $h\neq 0$, the necessary and sufficient condition for $z^{-m}F_{h,m}(z)=0$ to have imaginary roots is that $2m<-1$ and not be equal to an integer. Moreover, if $0<|m+p|<\frac12$, p a positive integer, then this equation has precisely $2p$ imaginary roots. Furthermore, whether h is zero or not, the amplitudes of two distinct imaginary roots are always distinct." *M. A. Basoco* (Lincoln, Neb.).

Chaundy, T. W. Expansions of hypergeometric functions. Quart. J. Math., Oxford Ser. **13**, 159–171 (1942). [MF 7915]

The following expansions are proved:

$$\begin{aligned} F(A, B; C; x) &= \sum_{r=0}^{\infty} (-1)^r \frac{(a)_r(b)_r}{r!(c)_r} {}_4F_3\left[\begin{matrix} A, B, c, -r \\ a, b, C \end{matrix}\right] x^r F(a+r, b+r; c+r; x) \\ &= \sum_{r=0}^{\infty} (-1)^r \frac{(a)_r(b)_r}{r!(c+r-1)_r} {}_4F_3\left[\begin{matrix} A, B, c+r-1, -r \\ a, b, C \end{matrix}\right] \\ &\qquad \times x^r F(a+r, b+r; c+2r; x), \end{aligned}$$

$$(a)_r=a(a+1)\cdots(a+r-1);$$

similar but naturally more complicated formulas hold for Appell's double hypergeometric functions. *G. Szegö.*

Burchnall, J. L. Differential equations associated with hypergeometric functions. Quart. J. Math., Oxford Ser. **13**, 90–106 (1942). [MF 7633]

The starting point of the author is the following remark which can be verified directly. Let b, b' be constants and let $z=z(p, q, x)$ satisfy the partial differential equation

$$\theta f(\theta+\varphi)z=px(\theta+b)g(\theta+\varphi)z,$$
$$\varphi f(\theta+\varphi)z=qx(\varphi+b')g(\theta+\varphi)z,$$

$\theta=p\partial/\partial p$, $\varphi=q\partial/\partial q$; then z is a solution of the ordinary differential equation

$$\{\delta f(\delta)f(\delta-1)-x[p(\delta+b)+q(\delta+b')]f(\delta)g(\delta) + pqx^2(\delta+b+b')g(\delta)g(\delta+1)\}z=0, \quad \delta=x\partial/\partial x.$$

As a first application, a differential equation of the third order is obtained for Appell's function $F^{(1)}[a; b, b'; c; px, qx]$. This equation can be solved completely in terms of functions $F^{(1)}$ (apart from certain degenerated cases like $p=q$). Furthermore, an equation of the fifth order is derived with the solutions $J_{\pm\mu}(ax)J_{\pm\nu}(bx)$. Finally differential equations are constructed which are satisfied by the product of two Whittaker functions. Various extensions and particular cases are considered. *G. Szegö.*

Lowan, Arnold N. and Horenstein, William. On the function $H(m, a, x)=\exp(-ix)F(m+1-ia, 2m+2; 2ix)$. J. Math. Phys. Mass. Inst. Tech. **21**, 264–283 (1942). [MF 8173]

Starting from the differential equation satisfied by the function $H(m, a, x)$ defined in the title the authors derive an integral representation and various recurrence formulas. Numerical values of H and its derivative are tabulated to seven significant figures for integral values of the three parameters: $0\leqq x\leqq 10$, $0\leqq a\leqq 10$, $0\leqq m\leqq 3$. These values are computed from a series expansion and checked by using the recurrence formulas. *M. C. Gray.*

Basoco, M. A. On the Fourier developments of a certain class of theta quotients. Bull. Amer. Math. Soc. **49**, 299–306 (1943). [MF 8228]

Fourier developments are obtained for the functions $\varphi_a^k(z)=\{\vartheta_a'(z,q)/\vartheta_a(z,q)\}^k$ (where $\vartheta_a(z,q)$ stands for one of the four Jacobi theta functions, k for a positive integer), the Fourier coefficients of φ^k being given in terms of those of φ^s $(s=1,\cdots,k-1)$ by a recurrence relation of order k. Obvious identities yield paraphrases of Liouville-Bell. Contacts with articles of Uspensky, Nichols and Glaisher are indicated. *A. B. Coble* (Urbana, Ill.).

Differential Equations

Kasner, Edward. Differential equations of the type $y^{iv}=Ay'''^2+By'''+C$. Univ. Nac. Tucumán. Revista A. **3**, 7–12 (1942). [MF 8145]

This note discusses various problems in geometry and physics which give rise to differential equations of the type $y^{(iv)}=Ay'''^2+By'''+C$, in which A, B and C are functions of x, y, y' and y''. The topics treated include systems of osculating conics to a curvature element, systems of extremals and various systems of curves related to the trajectories in a force field. *P. Franklin.*

Chiellini, Armando. Ancora sugli invarianti del sistema formato da due equazioni differenziali lineari del secondo ordine e su classi di sistemi riducibili a coefficienti costanti. Pont. Acad. Sci. Comment. **6**, 525–554 (1942). [MF 7471]

L'auteur suppose que le système soit réduit préalablement à une des deux formes normales suivantes:

$$(1)\qquad \begin{aligned} y''+p_{12}z'+q_{11}y+q_{12}z=0,\\ z''+p_{21}y'+q_{21}y+q_{22}z=0, \end{aligned}$$

ou bien

$$(2)\qquad \begin{aligned} y''+p_1z''+q_{11}y+q_{12}z=0,\\ y'+p_2z'+q_{21}y+q_{22}z=0. \end{aligned}$$

Pour chacune il établit un système complet de 6 invariants relatifs et par conséquence 5 relatifs, par rapport aux substitutions de la forme

$$y=\lambda\bar{y},\quad z=\mu\bar{z},\quad \xi=\varphi(x).$$

Dans le cas du premier système les 6 invariants relatifs se groupent en deux groupes correspondants respectivement aux deux équations. Selon que les coefficients sont quelconques ou bien certains d'entr'eux sont nuls, ces invariants prennent des formes légèrement différentes. L'auteur étudie en particulier le cas des systèmes réductibles aux coefficients constants, entre lesquels une classe particulière qu'il appelle systèmes fuchsiens et pseudofuchsiens de première espèce. *B. Levi* (Rosario).

Riblet, Henry J. Symmetric differential expressions. Bull. Amer. Math. Soc. **48**, 871–873 (1942). [MF 7503]

The present paper gives an improved procedure for expressing a symmetric differential polynomial in terms of the elementary symmetric functions, their derivatives and certain negative powers of the discriminant D [cf. Riblet, Amer. J. Math. **63**, 339–346 (1941), in particular, p. 341; these Rev. **2**, 346]. The present method depends on a lexicographic ordering of terms and yields an expression in which the power of D^{-1} used is least. *N. Jacobson.*

Langer, Rudolph E. A theory for ordinary differential boundary problems of the second order and of the highly irregular type. Trans. Amer. Math. Soc. **53**, 292–361 (1943). [MF 8124]

This paper studies the second order differential system composed of a second order equation with coefficients dependent upon the independent variable and a complex parameter together with two point linear boundary conditions. The coefficients of the differential equation are at most quadratic in the parameter while those of the boundary conditions may be any polynomials in the parameter. This system is shown to normalize into one of the somewhat more general type:

$$u'(x)=[\lambda P(x)+Q(x)]u(x),\quad H_a(\lambda)u(a)+H_b(\lambda)u(b)=0,$$

where capital letters denote matrices of order two and $u(x)$ is a vector of two components, and the paper actually discusses this matrix system. In the past, boundary value problems for such systems have been classified as "regular," "mildly irregular" and "highly irregular." The expansion theory for "regular" and "mildly irregular" systems has been developed in fairly complete form but the methods used for such systems seem to lead to serious difficulties when attempts are made to apply them to highly irregular problems. For the latter type, only special problems have been attacked and the results for these have been rather limited in character.

The present paper introduces a new approach to highly irregular problems through imbedding any such problem in a continuous family of regular boundary problems and then using limiting processes to carry over results from existing theories. When a problem is so imbedded, it is classified into one of two categories, A or B, according as it is or is not possible to set up a one to one correspondence between the characteristic values of the irregular system and those of the regular systems of the imbedding family in such a way that the former characteristic values are all limiting values of those of the latter systems. For systems in category A, a rather complete expansion theory is developed which follows the lines of the theory for regular and mildly irregular systems; the paper thus brings this category of problems to a solution stage comparable to that occupied by these other systems. Category A contains all of the special irregular systems previously investigated. Category B does not yield to the methods of the paper and the author expresses the opinion that "it seems improbable that any expansion properties as conventionally understood inhere in problems of this type." *W. M. Whyburn.*

Langer, R. E. What are Eigen-werte? Amer. Math. Monthly **50**, 279–287 (1943). [MF 8329]

An expository article.

Heins, Albert E. A mixed boundary value problem. Some remarks on a problem of A. Weinstein. Bull. Amer. Math. Soc. **49**, 130–133 (1943). [MF 7994]

The author considers the two-dimensional functions u, harmonic over the strip $0<y<1$, satisfying the boundary conditions $u(x,0)=0$, $u(x,1)=ku_y(x,1)$, and which are either bounded or of finite exponential order for $-\infty<x<\infty$. The boundary condition at $y=0$ is handled by negative reflection across $y=0$. By applying the Laplace transform with finite limits in the form

$$g(x,s)=\int_{-1}^{1}e^{-sy}u(x,y)dy,$$

representations are obtained for u which were previously obtained by Cooper [J. London Math. Soc. 14, 124–128 (1939)] by means of contour integrals. *H. Poritsky.*

Kuznetsov, E. S. Conditions for heat flows on the boundary surface of two media, radiating heat transfer being taken into account. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1942**, 243–248 (1942). (Russian. English summary) [MF 8089]

The author derives the condition which has to be satisfied on a surface which divides two media with different coefficients of conductivity. He assumes that in addition to the heat transfer a flow of radiation takes place. Some special cases are discussed in detail. *S. Bergman.*

Rose, M. E. The specular reflection of plane wave pulses in media of continuously variable refractive properties. Phys. Rev. (2) **63**, 111–120 (1943). [MF 8046]

For a nondissipative, nondispersive medium finitely stratified in the x direction, the propagation of a plane monochromatic wave is given by

$$\phi_\nu = A_\nu \exp k_\nu z_\nu + B_\nu \exp k_\nu z_\nu', \qquad B_{n+1}=0, \tag{1}$$

with $z_\nu = x\cos\theta_\nu + y\sin\theta_\nu$, $z_\nu' = -x\cos\theta_\nu + y\sin\theta_\nu$, $k_\nu = w/c_\nu$, $\nu = 0, \cdots, n+1$, subject to (2) $\partial\phi/\partial x$, $\rho\phi$ continuous at the interfaces. Thus A_ν, B_ν are determined in terms of A_0 and Snell's law is satisfied. Write

$$\rho_\nu \tan\theta_\nu/(\rho_{\nu+1}\tan\theta_{\nu+1}) = 1+\epsilon_{\nu+1} \tag{3}$$

and assume below that higher powers of ϵ_ν are negligible. With

$$\phi_\nu = \sum_1^\nu (\delta_j \cos\theta_j)/c_j, \tag{4}$$

where δ_j is the thickness of the jth stratum, the reflection coefficient is

$$r = B_0/A_0 = -\tfrac{1}{2}\sum_1^{n+1} \epsilon_\nu \exp 2iw\phi_{\nu-1}. \tag{5}$$

The writer introduces a Fourier integral argument to determine the reflection for a pulse, but the result is clear, for, since c_ν is frequency independent and only single reflections are considered, the reflected wave is the resultant of $n+1$ pulses, each differing from the original merely by an obvious attenuation factor and a phase change. For a continuous medium the writer uses a heuristic argument to replace (3) by

$$\epsilon(x) = (d/dx)\log(\rho\tan\theta)dx,$$

(4) by

$$\phi(x) = \int_0^x \cos\theta dx/c$$

and (5) by

$$r = \tfrac{1}{2}\int_0^\infty (d/dk)\log(\rho\tan\theta)e^{2iw\phi(x)}dx$$

(the writer's expression omits w). A practical approximation assumes θ, c constant for (4). Application is made to reflection of pulses in sea water and in air. *D. G. Bourgin.*

Beckenbach, E. F. and Reade, Maxwell. Mean-values and harmonic polynomials. Trans. Amer. Math. Soc. **53**, 230–238 (1943). [MF 8121]

Beginning with the known results that the class of summable functions harmonic in the interior of a finite domain D is characterized by either of the properties that the value of $f(x, y)$ at each point (x_0, y_0) is equal (a) to its mean-value on the disc $D(x_0, y_0; r)$ of every circle interior to D with center (x_0, y_0), (b) to its mean-value on the circumference $C(x_0, y_0; r)$ of every such circle, the authors prove that if $f(x, y)$ is continuous then a necessary and sufficient condition that $f(x, y)$ be harmonic in D is that these two mean-values be identical. The main part of the paper is devoted to a study of corresponding areal and peripheral mean-values in which circles are replaced by n-gons $p_n(x_0, y_0; r; \varphi)$, in which r is the apothegm and φ is the angle of orientation. Results analogous to those mentioned above are obtained and it is shown that the mean-value property (areal or peripheral) is characteristic of classes of elementary harmonic functions, namely, harmonic polynomials of degree at most $n-1$ or n, according as the parameter φ is unrestricted or constant. A representative result is the following. If $f(x, y)$ is superficially summable in the interior of a finite domain D, and if n is a fixed integer, $n \geqq 3$, then a necessary and sufficient condition that, at each point (x_0, y_0) in D, the value of $f(x, y)$ be equal to its mean-value on the interior of every n-gon $p_n(x_0, y_0; r, \varphi)$ in D is that $f(x, y)$ be a harmonic polynomial of degree at most $n-1$. Reference is made to the work of J. L. Walsh [Bull. Amer. Math. Soc. 42, 923–930 (1936)] who obtained results, similar to some of these, for means of the n values of a function at the vertices of a regular n-gon. *H. E. Bray.*

Loomis, Lynn H. The converse of the Fatou theorem for positive harmonic functions. Trans. Amer. Math. Soc. **53**, 239–250 (1943). [MF 8122]

Let the harmonic function $v(z)$ be represented by the Poisson-Stieltjes integral

$$v(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} dV(\theta),$$

where $V(\theta)$ is of bounded variation. The Fatou theorem states: (A) if $\lim_{t\to 0}(V(\theta+t)-V(\theta-t))/2t$ exists then $v(z)$ approaches the same limit along the corresponding radial path; (B) if $V(\theta)$ has a derivative at some point then $v(z)$ approaches $V'(\theta)$ as z approaches this point along any chord. The author works in the half-plane instead of the unit circle, where, of course, there are entirely analogous theorems. It is shown that the converse statements are true in the case of positive harmonic functions. In this case the converse of (A) is a consequence of a Tauberian theorem of Hardy and Littlewood. The converse of (B) is proved by repeated application of the converse of (A) and at the same time opening up an angular region into a half-plane. More general results are proved. An example is given which shows that the unrestricted converses of (A) and (B) are false.

A. C. Schaeffer (Stanford University, Calif.).

Calculus of Variations

Blankinship, W. A. The curtain rod problem. Amer. Math. Monthly **50**, 186–189 (1943). [MF 8167]

Using the methods of the calculus of variations, the following problem is solved in detail: determine the shape that a rod of uniform cross-section, density and elasticity will take if forced to pass through the three noncollinear points

$(a, 0)$, $(-a, 0)$ and $(0, b)$. The parametric equations defining the shape of the rod are well-known functions when arc length is used as a parameter. *F. G. Dressel.*

***Bobonis, Augusto. Differential systems with boundary conditions involving the characteristic parameter.** Contributions to the Calculus of Variations, 1938–1941, pp. 99–138. University of Chicago Press, Chicago, Ill., 1942. $3.00.

This paper is a direct extension of the work of Bliss [Trans. Amer. Math. Soc. **28**, 561–584 (1926); **44**, 413–428 (1938)] on definitely self-adjoint systems of linear differential equations with linear boundary relations. Such a system arises from the study of the second variation in a Bolza problem in the calculus of variations. The system of equations is

$$y'(x)=[A(x)+\lambda B(x)]y(x), \qquad a\leqq x\leqq b,$$

subject to $(M_0+\lambda M_1)y(a)+(N_0+\lambda N_1)y(b)=0$, where $y(x)$ is a vector and $A(x)$, $B(x)$, M_0, M_1, N_0, N_1 are square matrices. The system is self-adjoint if the adjoint system is equivalent to the original system under a transformation $z(x)=T(x)y(x)$; it is definitely self-adjoint if additional conditions which involve the symmetry and positive definiteness of a certain quadratic expression are satisfied. As in Bliss [loc. cit.], the usual eigenvalue theory, including the expansion theorem, can be developed for definitely self-adjoint systems. *M. Shiffman* (New York, N. Y.).

***Mansfield, Ralph. Differential systems involving k-point boundary conditions.** Contributions to the Calculus of Variations, 1938–1941, pp. 413–452. University of Chicago Press, Chicago, Ill., 1942. $3.00.

This paper considers a system of first order differential equations, where the solutions are permitted to have jump discontinuities at $k-2$ interior points, subject to linear boundary conditions involving these $k-2$ points and the two end points. A simple transformation changes the problem to an ordinary two-point system, and the known theory of this case entails analogous results for the k-point problem. In particular, the possibility of continuous solutions satisfying the k-point boundary conditions is discussed, and the existence of such solutions made to depend on simpler properties of the adjoint differential system. An application is made to similar problems in the complex plane.

M. Shiffman (New York, N. Y.).

Myers, Franklin G. Sufficiency conditions for the problem of Lagrange. Duke Math. J. **10**, 73–97 (1943). [MF 8103]

The present paper is concerned with obtaining a sufficiency theorem for the nonparametric problem of Lagrange under weaker hypotheses than has been used heretofore. It is shown that an arc C_0 of class C'' affords a semi-strong relative minimum to the integral J under consideration. For each differentiable admissible variation of $\eta\neq 0$ there is a set of multipliers λ^0, $\lambda^\beta(x)$ with which C_0 satisfies the Euler-Lagrange equations, the strengthened conditions of Weierstrass and Clebsch and $J_2(\eta, \lambda)>0$, where $J_2(\eta, \lambda)$ is the second variation of J along C_0. It is shown that for certain integrands the arc C_0 will afford a strong relative minimum to J under the above hypotheses. The method used is an extension of an indirect method used by McShane [Trans. Amer. Math. Soc. **52**, 344–379 (1942); these Rev. **4**, 48] in order to establish sufficiency theorems for weak relative minima. *M. R. Hestenes* (Chicago, Ill.).

Mammana, Gabriele. Il minimo assoluto in taluni classici problemi di calcolo delle variazioni. I, II. Anais Acad. Brasil. Ci. **14**, 53–77, 167–185 (1942). (1 plate) [MF 7258]

The problem of determining the absolute minimum of the functional

$$J_L=\int_{P_1}^{(L)\,P_2} y^{1/n}(dx^2+dy^2)^{\frac{1}{2}},$$

n arbitrary, L in the upper half-plane, is given a simple and elementary treatment. For $n=-2, -1, 0, 1, 2$, the problem has familiar geometrical interpretations (brachistóchrone for $n=-2$, surface of revolution of minimum area for $n=1$, etc.); the results obtained for $n<0$ and for $n\geqq 1$ are analogous to the well-known results in these special cases.

The Euler equation for the extremals of J_L is

$$nyy''=1+y'^2, \qquad n\neq 0,$$

the integrals of which are called curves of Ribaucour. These curves are studied from the viewpoint of differential calculus. In part I, the discussion of relative minima, given for a general n, suffices to show, for $n<0$, that a curve of Ribaucour is the unique minimizing arc joining P_1, P_2, and therefore gives an absolute minimum. For a fixed $n>0$ and for a fixed $P_1:(x_1, y_1)$, the curves of Ribaucour have an envelope Γ. For $n\geqq 1$, Γ extends from $(x_1, 0)$ to $(+\infty, +\infty)$ but, for $0<n<1$, Γ has a vertical asymptote and extends from $(x_1, 0)$ to $(x_1+r, +\infty)$, $0<r<+\infty$. Let E be the domain in the upper half-plane bounded by $x=x_1$ and Γ, and let $x_2>x_1$. Then, for $n\geqq 1$, there is always a $y_2>0$ sufficiently large that there is a curve of Ribaucour through P_1 and $P_2:(x_2, y_2)$, but this is not necessarily true for $0<n<1$. It is shown in part II that, if a point P_2 is contained in E, then the only minimizing curves are a curve of Ribaucour and the broken-line solution of Goldschmidt, so that one of these furnishes the absolute minimum; if P_2 is not in E, then the Goldschmidt solution gives the only relative minimum and therefore the absolute minimum. It is proposed, in a forthcoming part III, to determine for points in E which of the two relative minima is the absolute minimum.

E. F. Beckenbach (Austin, Tex.).

Courant, R. Variational methods for the solution of problems of equilibrium and vibrations. Bull. Amer. Math. Soc. **49**, 1–23 (1943). [MF 7877]

This article is a survey of a class of boundary value problems which can be formulated as variational problems. Five aspects of such problems are discussed with special emphasis on approximation methods. (a) There is a brief discussion of the minimizing (or rendering stationary) of quadratic functionals which arise in elasticity, and the rigid and natural boundary conditions associated with such problems. There are some general physical remarks which will be useful for the practical solution of these problems. (b) This introduction is followed by a brief discussion of the Rayleigh-Ritz method with remarks on such pertinent points as the choice of sequence of functions, "sensitizing functionals" and estimation of errors by this method. (c) Method of finite differences [Phillips and Wiener, J. Math. Phys. Mass. Inst. Tech. **2**, 105–124 (1923); Courant, Friedrichs and Lewy, Math. Ann. **100**, 32–74 (1928)]. (d) Hadamard's Method of Gradients. This has been discussed by Courant [Theodore von Kármán Anniversary Volume, pp. 189–194, California Institute of Technology, Pasadena, Calif., 1941; these Rev. **2**, 368]. (e) Finally there is a brief discussion of the numerical treatment of the plane

torsion problem for multiply-connected regions. A bibliography of the more important papers is appended.

A. E. Heins.

Threlfall, W. Le calcul des variations global. Enseignement Math. **38**, 189–208 (1942). [MF 7294]

The present paper is descriptive in character. Its purpose is to describe in an elementary fashion the ideas underlying the theory of critical points of functions in the large. The general problem of critical points is formulated. However, the results are brought out by means of examples of elementary nature. The development follows closely the ideas presented by H. Seifert and W. Threlfall [Variationrechnung im Grossen, Teubner, Leipzig, 1938]. The paper forms an excellent introduction to the theory of critical points as developed by M. Morse. *M. R. Hestenes* (Chicago, Ill.).

Goldstine, H. H. The calculus of variations in abstract spaces. Duke Math. J. **9**, 811–822 (1942). [MF 7931]

The author considers the problem of minimizing an integral

$$I=\int_{x_1}^{x_2} f[x, y(x), y^1(x)]dx,$$

where, for each x, $y(x)$ is a point of a Banach space $\mathfrak{B}$ and $f[x, v, w)$ is a real-valued function whose arguments v and w lie in the space $\mathfrak{B}$. Sets of necessary conditions and of sufficient conditions for a minimum are obtained which generalize those of the classical theory for n-dimensional space. The space $\mathfrak{B}$ is restricted to be a space of real-valued functions defined on an arbitrary class $\mathfrak{P}$ of elements p, and it is assumed that there exists a fundamental set $\mathfrak{N}$ in $\mathfrak{B}$ whose cardinal is not greater than that of $\mathfrak{P}$. The fulfillment of the condition of nonsingularity described at the bottom of page 813 seems to depend on the proper choice of this fundamental set $\mathfrak{N}$ and of a correspondence between $\mathfrak{N}$ and $\mathfrak{P}$. There also seems to be implicit in the theory some restriction on the type of norm in the space $\mathfrak{B}$.

L. M. Graves (Chicago, Ill.).

Mathematical Biology

Wilson, Edwin B. and Burke, Mary H. The epidemic curve. I, II. Proc. Nat. Acad. Sci. U. S. A. **28**, 361–366 (1942); **29**, 43–48 (1943). [MF 7682]

If C_i is the number of infectious cases and S_i the number of susceptibles in the ith generation, A the number of susceptibles per generation coming into the population and m a proportionality factor, Soper's finite difference equations are (1) $C_iS_i=mC_{i+1}$; (1′) $S_i+A=C_{i+1}+S_{i+1}$. The authors consider, instead of (1),

$$(2)\qquad C_{i+1}/S_i=1-(1-S_i^{-1})^{rC_iS_i},$$

which takes into account the possibility of multiple contacts between a susceptible and the infectious persons and eliminates the absurd conclusion $C_i>S_i$ to which (1), (1′) sometimes lead. In (2), r is a proportionality factor (the "contact rate"). If C_i and r are small, (2) reduces approximately to (1) with $r=1/m$. If S_i is not too small, (2) may be replaced by $C_{i+1}/S_i=1-\exp(-rC_i)$; for $A=0$ one has (*) $S_k=S_0\exp(rS_k-rS_0)$, which is tabulated in the form $S_k=f(r, S_0)$ and leads to the conclusion that S_k is independent of k. [Since $A=0$, S_k should be a decreasing function of k. The contradiction may be eliminated by assuming r variable from generation to generation; this gives a formula like (*) with r replaced by $\rho_k=\sum r_iC_i/\sum C_i$, the summation extending over the first k generations.] The authors consider next Soper's differential equation $md^2 \log z/dt^2=A_1-z$, where z and A_1 are the time rates of infectious cases and of new susceptibles, respectively. The period (twice the time interval between the maximum and the minimum value of z) is calculated. *I. Opatowski* (Chicago, Ill.).

Pitts, Walter. A general theory of learning and conditioning. I. Psychometrika **8**, 1–18 (1943). [MF 8142]

It is assumed that, of M possible, exclusive and dissimilar reactions, a given organism will react with the jth when the jth "simple response-tendency" $E_j(t)=S_j(t)+Q_j(t)$ exceeds a certain threshold. This simple response-tendency consists of an unconditioned part $S_j(t)$ and a conditioned part. Of N dissimilar types of stimuli, $P_i(t)$ measures the intensity of the ith at time t. Then $T_{ij}^0(\delta, t)$ being a measure of the contribution of unit $P_i(t)$ to $S_j(t+\delta)$, it follows that

$$S_j(t)=\int_0^t \sum P_i(\theta)T^0_{ij}(t-\theta, \theta)d\theta.$$

Correspondingly,

$$E_j(t)=\int_0^t \sum P_i(\theta)T_{ij}(t-\theta, \theta)d\theta.$$

A nonzero $E_j(\eta)$ following a positive $P_i(\xi)$ leads to an increment (possibly negative) in $T_{ij}(\delta, t)$, with a Gaussian distribution as to δ about $\eta-\xi$, decaying as δ increases, and decaying also with time. This increment is thus

$$\Delta_{\xi\eta}T_{ij}(\delta, t)d\xi d\eta=P_i(\xi)[S_j(\eta)-Q_j(\eta)] \times\exp\{-\alpha\delta-\beta^2(\eta-\xi-\delta)^2-\gamma(t-\eta)\}d\xi d\eta.$$

After making certain substitutions and simplifications, the author arrives at equations of the form

$$Q_j(t)=\int_0^t F(\eta, t)[S_j(\eta)-Q_j(\eta)]d\eta$$

for the determination of the Q_j.

A further section considers the phenomenon of extinction, the assumption being that the reaction thresholds $R_j(t)$ vary in a specified manner with the course of the conditioning, and explicit formulas for these are deduced involving quadratures. The arguments of the T's and ΔT's as stated do not correspond. Also, the definition of T^0 omits a necessary time factor. *A. S. Householder* (Chicago, Ill.).

Koyenuma, Nobutsugu. Beiträge zur Theorie der biologischen Strahlenwirkung. Z. Phys. **120**, 185–211 (1943). [MF 8246]

Let uni-cellular objects be exposed to harmful radiation. As usual, it is assumed that the probability that an injured object has been hit by at least n quanta is given by the Poisson expression

$$(1)\qquad f(D)=1-e^{-\alpha D}\sum_{k=0}^{n-1}(\alpha D)^k/k!,$$

where α is a probability parameter and D the dose. The behavior of $f(D)$, in particular, its point of inflection, is studied in detail using estimates given by Pólya and Szegö [Aufgaben und Lehrsätze aus der Analysis, Springer, Berlin, 1925, vol. 1, page 80]. The formulae involve Bessel functions. The theory is generalized in various directions. For example, it is assumed that $D=It^p$, where I is the intensity of the radiation and t the time of exposure (Schwarzschild's law). Another generalization consists in assuming that the sensitivity of the cell changes with the number of quanta by which it has been hit. This leads, under certain assump-

tions, to expressions of the form

(2) $1-e^{-(x+y)}\{x+(1+y)x^2/2!+(1+y+y^2/2!)x^3/3!+\cdots\}$,

which are again studied in detail. Finally, the theory is generalized to multi-cellular objects. *W. Feller.*

Householder, Alston S. Cellular forms: the tri-axial cell. I. Bull. Math. Biophys. 4, 159–168 (1942). [MF 7525]

The approximate theory of Rashevsky [Advances and Applications of Mathematical Biology, University of Chicago Press, Chicago, Ill., 1940; these Rev. 2, 138], in which the shape of the cell is assumed of a spheroidal type, is extended to the cells whose shape is of the type of a parallelepiped or an ellipsoid. The hypotheses concerning the concentration inside the cell and the diffusion characteristics outside the cell are also less restrictive than in Rashevsky's theory. The equations of diffusion in the cell remain substantially unchanged by these generalizations. The equations of cell deformation are of the type $d\log r_i/dt=f_i(r_1, r_2, r_3)$, $i=1, 2, 3$, where r_i's are certain lengths representing the size of the cell. The equilibrium ($f_i=0$) and stability conditions are discussed in detail. *I. Opatowski.*

Pitts, Walter. The linear theory of neuron networks: the static problem. Bull. Math. Biophys. 4, 169–175 (1942). [MF 7526]

A neuron, on receiving at its "origin" a supra-liminal stimulation, is set into activity and produces at its "terminus" a state capable of acting as a stimulus or an inhibitor to the activity of any other neuron which originates there. In the steady state, this produced state has a measure which is a monotonic function of the stimulus intensity with an asymptote. The problem of the network is that of ascertaining, from the structure of the network and the activity functions of the individual fibers comprising it, which fibers will become active under any given pattern of externally-applied stimulation. The problem has been discussed previously by the reviewer and the author with solutions for special cases. Here the author introduces certain variables capable of assuming the values zero and unity, and transforms the problem to that of choosing those values of these variables which satisfy certain multilinear inequalities.
A. S. Householder (Chicago, Ill.).

Landahl, H. D. Equilibrium shapes in non-uniform fields of concentration. Bull. Math. Biophys. 4, 155–158 (1942). [MF 7524]

Following Young [Bull. Math. Biophys. 1, 31–46 (1939)], an analogue of Betti's theorem is used to express the rate of elongation or compression of a cell, this rate being taken as linear in the concentration. An exact form for this rate is deduced for the cases in which the concentration at any point of the cell membrane is expressible in terms of spherical harmonics, independently of the "longitude." This includes the case of two similar neighboring cells if the compression of only one of these by the other is considered. The permeability is taken as infinite. *A. S. Householder.*

NUMERICAL AND GRAPHICAL METHODS

***Thompson, A. J. Table of the Coefficients of Everett's Central-Difference Interpolation Formula.** Tracts for Computers, no. 5, 2nd edition. Cambridge University Press, Cambridge, England, 1943. viii+32 pp. 5 s.

The principal table, Table I, gives to ten places of decimals the values of the first four coefficients (that is, the coefficients of the 2nd, 4th, 6th and 8th central differences) in Everett's central-difference interpolation formula. Entries are given at intervals of 0.001 in the argument for the range 0 to 1 and second central differences are tabulated to facilitate interpolation. The table is so arranged that the values for the argument θ and for the argument $1-\theta$ may be read without turning pages. Table II gives exact values of the coefficients at intervals of 0.01, together with their 2nd, 4th, 6th and 8th central differences, while Table III gives the same at intervals of 0.1. Table IV is the same as Table III except that the range is from -5 to 6, so that it may be used for interpolation at the beginning or end of a table. The introduction describes in detail the construction of the tables, their use in direct and inverse interpolation, subtabulation, etc., and the procedure for the case where printed differences are not available. *W. E. Milne.*

Neuschuler, L. On double and triple term tables of functions of three variables. C. R. (Doklady) Acad. Sci. URSS (N.S.) 36, 121–124 (1942). [MF 8043]

The author discusses the problem of tabulating the values of a function $f(x, y, z)$ of three variables. To avoid a large triple entry table one may seek to represent f by one of the forms

(1) $$f(x, y, z)=\psi(\phi(x, y), z),$$

(2) $$f(x, y, z)=F(G(H(x, y), z), x).$$

Necessary and sufficient conditions for each of these representations in terms of partial differential equations which f must satisfy are given. In the first case one replaces the large triple entry table by two small double entry tables I and II. The most convenient arrangement is that in which Table I gives inversely the function $y=\phi^{-1}(x, \phi)$ and table II gives $f=\psi(\phi, z)$, the two tables having the common argument ϕ. For (2) the author suggests the three double entry tables giving $y=H^{-1}(x, H)$, $z=G^{-1}(H, G)$ and $f=F(G, x)$. *D. H. Lehmer* (Berkeley, Calif.).

Running, T. R. Graphical solutions of cubic, quartic, and quintic. Amer. Math. Monthly 50, 170–173 (1943). [MF 8163]

Hotelling, Harold. Some new methods in matrix calculation. Ann. Math. Statistics 14, 1–34 (1943). [MF 8248]

This is a very useful exposition of modern methods of solving linear equations, computing determinants, the inverse of matrices, etc. The advantages and disadvantages of each method are carefully explained and illustrative examples are given. Special attention is given to iterative methods, and various devices of accelerating the convergence are described. Many improvements are due to the author. The delicate problem of estimating the error involved is also extensively discussed along new lines. The methods of computing characteristic roots and vectors are also briefly touched on. For details, we must refer to the paper. *W. Feller* (Providence, R. I.).

Bickley, W. G. Formulae relating to Bessel functions of moderate or large argument and order. Philos. Mag. (7) 34, 37–49 (1943). [MF 8018]

"Search of extant literature for formulae to aid in the computation of the zeros of the derivatives of Bessel func-

tions revealed large gaps in the available information, and it became necessary to derive the desired formulae, particularly those corresponding to the various asymptotic expansions for the functions themselves. The investigation brought to light further interconnections between formulae for the derivates and those for the functions, and some new results for the functions; it has therefore seemed desirable to give a comprehensive set of formulae, even if this involves the repetition of a few well-known ones. The aim is purely utilitarian, in the sense that the formulae are needed as bases for computation, and that explicit developments to a number of terms, rather than analytical proofs of the existence of such developments, are therefore essential. Elementary methods have been used, and such processes as the term-by-term differentiation of asymptotic expansions freely employed."

Following this introductory statement the author derives McMahon's expansion for the zeros of the Bessel functions and for the zeros of their derivatives and obtains formulae for the values of the derivatives at the zeros of the functions and for the values of the functions at the zeros of the derivatives. For the case where x/n is less than, or not much greater than, unity he employs the Debye expansions. Numerical values of the coefficients in these various expansions are given up to an order deemed sufficient for purposes of calculation. *W. E. Milne* (Corvallis, Ore.).

Behrbohm, Hermann. Kurze Bemerkung zur graphischen Lösung gewöhnlicher linearer Differentialgleichungen 1. Ordnung. Z. Angew. Math. Mech. **22**, 57–58 (1942). [MF 7667]

The author observes that the general solution of the linear first order differential equation is linear in the dependent variable y and the constant of integration C. If the independent variable x is considered as a parameter to which one assigns various constant values, the general solution can, therefore, be represented by a level curve chart consisting of a family of straight lines of equal x in a plane with (y, C) coordinate axes. *P. W. Ketchum* (Urbana, Ill.).

Magnusson, Philip C. A numerical method of solving integral equations in two independent variables. J. Math. Phys. Mass. Inst. Tech. **21**, 250–263 (1942). [MF 8172]

The author presents and illustrates a method for the numerical solution of linear nonhomogeneous integral equations in two variables of the type

$$L(x, y)=L_0(x, y)+p(x, y)\int\int K(x, y; s, t)L(s, t)dsdt.$$

The procedure consists in replacing the double integral by a two dimensional Simpson rule (or higher degree Newton-Cotes rule) applied to the values of the integrand at the vertices of a rectangular net, so as to have one algebraic equation in the unknown values of L for each point, and finally solving these equations. The case where K has very large or even infinite values is discussed, and the simplifications due to symmetry are noted. Application is made to the problem of inter-reflections of light between two parallel square surfaces. *W. E. Milne* (Corvallis, Ore.).

Kantorovitch, L. V. Application of Galerkin's method to the so-called procedure of reduction to ordinary differential equations. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 31–40 (1942). (Russian. English summary) [MF 7733]

The Dirichlet problem for an elliptic partial differential equation $L(u)=0$ with the condition $u=\varphi(s)$ on the boundary is equivalent to the variational problem $I(u)=\min.$, where $I(u)$ is a quadratic functional. An approximate solution is obtained by putting

$$u_n(x, y)=\textstyle\sum_1^n\chi_k(x, y)f_k(x)+\chi_0(x, y),$$

where $\chi_k(x, y)$ are given functions vanishing on the boundary while χ_0 takes the prescribed values. The condition $I(u_n)=\min.$ yields a system of ordinary differential equations for the unknown functions $f_k(x)$. The author applies his method to domains which can be decomposed into rectangles. As an illustration, the torsion of an angle-iron is treated. The first publication of the procedure goes back to 1933 [Bull. Acad. Sci. URSS (7) **1933**, 647–652]. It should be noted that an identical procedure was suggested by H. Poritsky [Proc. Fifth Intern. Congress for Appl. Mech., Cambridge, Mass., 1938, p. 700]. *A. Weinstein* (Toronto, Ont.).

Pearlman, Y. I. Galerkin's method in calculus of variations and in the theory of elasticity. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 345–358 (1941). (Russian. English summary) [MF 7719]

This paper is a review of Galerkin's method and its applications to various problems. Let $u_n=u_0(x, y, z)+\sum_1^n c_k\varphi_k(x, y, z)$, where u_0 takes prescribed values on the boundary of a domain while all φ_k vanish. An approximate solution of the differential equation $E(u)=0$ with the prescribed boundary values is obtained, according to Galerkin's method, by replacing $E=0$ by the n equations $\int\int\int\varphi_k E(u_n)dxdydz=0$. The method is identical with the Ritz method in the case when $E(u)=0$ is the Euler equation of a variational problem. A list of papers referring to Galerkin's method is given, including several new Russian textbooks on elasticity. *A. Weinstein.*

Born, W. T. and Kendall, J. M. Application of the Fourier integral to some geophysical instrument problems. Geophysics **6**, 105–115 (1941). [MF 8313]

This paper considers only the simple part of Fourier integral theory required for the work in hand. A special type of Henrici rolling sphere harmonic analyzer has been built which makes possible a large reduction in the amount of labor involved in dealing graphically with Fourier integral operations. Examples of the use of this procedure as applied to several geophysical instrument problems are given. *Author's summary.*

Siegert, Arnold J. F. A mechanical integrator for the computation of gravity anomalies. Geophysics **7**, 354–366 (1942). [MF 8312]

MATHEMATICAL PHYSICS

Silberstein, L. Investigations on the intrinsic properties of the color domain. II. J. Opt. Soc. Amer. **33**, 1–10 (1943). [MF 7960]

[The first part appeared in the same J. **28**, 63–85 (1938).] In the two-dimensional domain of equiluminous colors the just perceptible chromaticity difference defines a line element. There is no a priori reason to assume this line element to be quadratic, but this assumption has been made by Helmholtz, Schrödinger and the author [see part I]. However, recent experiments by D. L. MacAdam [J. Opt. Soc. Amer.

32, 247–274 (1942)] show that this assumption is correct. The present paper is concerned with the geometry of the color surface defined intrinsically by the line element provided by MacAdam's experiments. The Gaussian curvature is calculated. It is found to take both positive and negative values, and to be of significant magnitude, giving a defect as great as 12 degrees in a triangle forming only 1/50 of the whole color domain. *J. L. Synge.*

Herzberger, M. Direct methods in geometrical optics. Trans. Amer. Math. Soc. **53**, 218–229 (1943). [MF 8120]

The purpose of this paper is to give a straightforward method of tracing skew rays through an optical system of revolution. [The author claims that the procedure can be applied to a general system.] The z-axis being taken along the axis of the instrument, an initial ray is defined by four quantities x, y, ξ, η; here x, y are the coordinates of the intersection of the ray with the plane $z=0$ and ξ, η are its first two components (direction cosines multiplied by refractive index). Similarly, the final ray is defined by x', y', ξ', η'. The purpose of the method is to determine x', y', ξ', η' in terms of x, y, ξ, η. From the symmetry of the instrument, the transformation is necessarily of the form

$$x'=Ax+B\xi,\quad \xi'=Cx+D\xi,$$
$$y'=Ay+B\eta,\quad \eta'=Cy+D\eta,$$

where A, B, C, D are functions of $u_1=\frac{1}{2}(x^2+y^2)$, $u_2=x\xi+y\eta$, $u_3=\frac{1}{2}(\xi^2+\eta^2)$. These functions are conditioned by the fact that this is a contact transformation; consideration of the fundamental optical invariant (Lagrange bracket) leads to a set of three partial differential equations for A, B, C, D, together with $AD-BC=1$. The equations are given in several forms and certain special cases are considered.

For a single refracting spherical surface, if the planes $z=0$, $z'=0$ pass through the center of the sphere, we have $B=0$, $A=1/D$, and there are only two functions C, D to determine; these are given explicitly. It is then an easy matter to shift the planes on which x, y, x', y' are measured away from the center of the sphere. When this is done, we have a step-by-step technique for calculating the transformation for any instrument of revolution formed of spherical surfaces. The relation between the partial derivatives of A, B, C, D and the Seidel errors is given, and the paper ends with a brief comparison of the author's method with that of Hamilton. *J. L. Synge* (Toronto, Ont.).

Herzberger, M. A direct image error theory. Quart. Appl. Math. **1**, 69–77 (1943). [MF 8189]

In the previous paper reviewed above, the author developed four functions related by three differential equations whereby image coordinates are obtained explicitly from object coordinates in a rotationally symmetric optical system. In the present paper, he develops these functions into series and substitutes in modified forms of the earlier differential equations. Gaussian optics is then deduced and image error functions are set up. Seidel and fifth-order errors are obtained by substitution. The paper concludes with a demonstration that image errors vanish for a plane and with the derivation of the Seidel and fifth-order coefficients for the case of a single sphere. *P. Boeder* (Southbridge, Mass.).

Synge, J. L. The theory of the Schmidt telescope. J. Opt. Soc. Amer. **33**, 129–136 (1943). [MF 8116]

The author calculates Hamilton's T-function (the angle characteristic) for a "solid" Schmidt camera up to the members representing Seidel's aberrations, a solid Schmidt being a Schmidt camera where the space between mirror and correction plate is filled with glass. In other words, the camera consists of a single block of glass with the spherical reflecting surface on one side and the figured correcting surface on the other. This reduces the elimination problems to a single one, which Synge succeeds in solving approximately by means of a parametric method. Spherical and chromatic aberrations are especially discussed.

M. Herzberger (Rochester, N. Y.).

Glaser, Walter. Über elektronenoptische Abbildung bei gestörter Rotationssymmetrie. Z. Phys. **120**, 1–15 (1942). [MF 8236]

Um den beträchtlichen Mangel an Übereinstimmung zwischen der praktisch erreichten Auflösung des Elektronenmikroskops und der auf Grund der Öffnungsfehler und Beugung berechneten zu erklären, wird der Einfluss einer kleinen Abweichung von der Rotationssymmetrie des Abbildungsfeldes auf die Begrenzung des Auflösungsvermögens untersucht. Die elektrischen und magnetischen Abbildungsfelder mit zwei aufeinander senkrechten Symmetrieebenen werden bestimmt, die Differentialgleichungen für die achsennahen Elektronenbahnen aufgestellt und die Gesetze der elektronen-optischen Abbildung abgeleitet. Insbesondere wird der "axiale Astigmatismus" untersucht und für ein Magnetfeld mit glockenförmigen, axialen Feldverlauf und für einen bestimmten Unsymmetriegrad der Radius des "Kreises der kleinsten Verwirrung" sowie Dehnung und Scherung formelmässig und numerisch berechnet.

P. Boeder (Southbridge, Mass.).

Schelkunoff, S. A. A mathematical theory of linear arrays. Bell System Tech. J. **22**, 80–107 (1943). [MF 8067]

The radiation pattern of an array of equal-pattern and parallel oriented elemental radiators is the product of the elemental radiation characteristic and the space factor of the array. Therefore nondirective point sources are considered as elements, without loss of generality. They are arranged along a straight line with mutual distances assumed to be multiples of a finite length. In this case, the space factor of the most general linear array (as referred to field strength) can be written as "radiation polynomial" $|\sum_0^{n-1} a_\nu z^\nu|$, where $a_\nu = A_\nu \exp(i\vartheta_\nu)$, $z=\exp(i\psi)$, $\psi=kl\cos\theta-\vartheta$, n the number of elements, A_ν and ϑ_ν the relative amplitude and phase of νth radiator, $k=2\pi/\lambda$, λ the wavelength, l the greatest common divisor of distances, θ the directional (latitude) angle, ϑ the progressive phase difference. This formulation of the space factor in terms of a complex variable of modulus one proves superior to the conventional representation as a trigonometric sum, as it is easily manipulated by elementary algebraic methods. Both analysis and synthesis of arrays are treated in this fashion and many practical examples are given, particularly optimal directional characteristics and asymptotic rules. The coverage $\bar{\psi}=2kl$ of the unit circle is of fundamental importance; for $l>\frac{1}{2}\lambda$, it has to be considered as a "Riemann circle." Optimal designs are obtained with ease via the location of the zeros of the radiation polynomial. *H. G. Baerwald* (Cleveland, Ohio).

Weber, Ernst. Ultrashort electromagnetic waves. I. Electromagnetic theory. Elec. Engrg. **62**, 103–112 (1943).

This article is expository in nature. It contains a derivation of Maxwell's field equations and a treatment of the following topics: a uniform plane electromagnetic wave, energy transport of waves in conductors, skin effect and

boundary conditions between two media. It is the first one of a sequence of several articles that are being prepared for familiarizing electrical engineers with some of the concepts and problems occurring in connection with ultra high frequencies and their solution by means of Maxwell's equations.

H. Poritsky (Schenectady, N. Y.).

Grünberg, G. Suggestions for a theory of the coastal refraction. Phys. Rev. (2) **63**, 185–189 (1943). [MF 8136]

This half space $z>0$ is air, that $z<0$ sea water except for an island representing a simply-connected domain S in the plane $z=0$, which may, in particular, be the half plane $x>0$ (continent). The propagation of electromagnetic waves in this system is treated. Apart from the geometrical idealizations of flatness and depth along the coastline of the island (continent), it is necessary, in order to make the problem manageable, to use the approximate boundary conditions:

$$\left.\frac{\partial E}{\partial z}\right|_{z=0}=0 \qquad \text{over sea,}$$

$$\left.\frac{\partial E}{\partial z}\right|_{z=0}=\frac{2\pi i}{\lambda}\left[\frac{\epsilon\mu}{\epsilon_0\mu_0}\left(1+\frac{\sigma}{i\epsilon\omega}\right)\right]^{-\frac{1}{2}}E_z\Big|_{z=0} \qquad \text{over land,}$$

where λ is the wave length in air (vacuum), ϵ, μ, σ the dielectric constant, magnetic permeability and electrical conductivity of the soil in M.K.S. units, ϵ_0 and μ_0 the d.c. and m.p. of vacuum, ω the circular frequency. The associated idealizations are: infinite conductivity of the sea water, and $2\pi^2d^2\lambda^{-2}\ll 1$, where d is the depth of field penetration (skin effect constant) in the soil. (Both are reasonably well fulfilled for long and medium waves, but the latter, in general, not for short waves.) Application of Green's theorem then yields the integral equation

$$(E_z)_{z=0}=2(E_z^0)_{z=0}+\alpha\int_{(S)}(E_z)_{z=0}r^{-1}\exp(2\pi i r\lambda^{-1})df,$$

where E_z is the "primary" field which would obtain in infinite free space under the given distribution of primary currents. In the case of the coast line $S:x>0$ and harmonic variation along the coast: $(E_z)_{z=0}=f(x)e^{isy}$ (arbitrary distribution may be built up by a Fourier integral), this equation reduces to

$$F(x)=2f(\alpha)+i\pi\alpha\int_0^\infty F(\xi)H_0^{(1)}(4\pi^2\lambda^{-2}-s^2|x-\xi|)d\xi,$$

which can be solved by quadratures by the method of Hopf and Wiener. For a plane wave propagated from the sea in arbitrary direction θ, $F(\alpha)=2\exp(imx)\cdot\psi(x)$, where $\psi(x)$ varies slowly compared to $\exp(imx)$. For this case, an explicit approximate solution is obtained as a contour integral of the Bromwich type whose asymptotic value $\psi(x)\cong(\lambda\cos\theta)^{\frac{1}{2}}/(2\pi^2dx^{\frac{1}{2}})$ $(x\to\infty)$ describes the slow decay of the "refracted" wave whose direction of propagation coincides with that of the primary, at considerable distance from the coast. *H. G. Baerwald* (New York, N. Y.).

Condon, E. U. Principles of micro-wave radio. Rev. Modern Phys. **14**, 341–389 (1942). [MF 8058]

Principles underlying radio technique in the frequency range of approximately 0.3 to 300 kilo-megacycles per second or 1 mm to 1 m wavelength are reviewed. This range lies between the optical range where radiation can no longer be produced by man-made oscillators, but is due entirely to noncoherent superposition of radiations from atoms and molecules, and that of "classical" radio technique where field theory can largely be circumvented by conventional circuit theory dealing with lumped quantities. In the micro-wave range no such technique of evasion exists. Furthermore, flight time of electrons is comparable with the frequency cycle. This review does not, however, deal with the electronic techniques, but is devoted entirely to the field theory of "circuit elements" involved, namely, cavity resonators and transmission lines. The theory of the former is developed in the conventional way from the Maxwell equations by means of orthogonal wave functions. Rectangular, cylindrical (singly and doubly connected cross sections), spherical resonators and figures of revolution ("doughnuts") are treated, rigorously where possible, otherwise by approximation methods. Resonator losses are derived in the customary way, via skin effect. The chapter on transmission lines comprises the conventional two-conductive line theory, load diagrams, impedance transformation, ohmic and dielectric losses, support effects, chokes and by-passes, line section resonators and tapered lines. (Wave guides, single conductor or dielectric, are included under cylindrical cavity resonators.)

H. G. Baerwald (Cleveland, Ohio).

Malti, Michel G. and Golomb, Michael. Electric propagation on long lines terminated by lumped networks. I. Line initially at rest. J. Franklin Inst. **235**, 41–73 (1943). [MF 7791]

Malti, Michel G. and Golomb, Michael. Electric propagation on long lines terminated by lumped networks. II. Line initially not at rest. J. Franklin Inst. **235**, 101–118 (1943). [MF 7918]

Solutions are obtained which are not restricted to the steady-state condition but which include transients, with all four line-parameters present, and with the line terminated in arbitrary, lumped networks. The Laplace transform method is employed. The particular contribution of this paper, intended to adapt the solution to numerical computation, consists in expanding the transformed solution in powers of a parameter, and inverting the resulting series term-by-term. The coefficients of the series thus obtained are repeated convolution integrals. It is suggested that these integrals be evaluated numerically, preferably by some such mechanical device as the cinema integraph [H. L. Hazen and G. S. Brown, J. Franklin Inst. **230**, 19–44, 183–205 (1940); these Rev. **2**, 62]. Furthermore, it is claimed that such a procedure leads to a method of solution which is readily evaluated numerically by persons not possessing any particular mathematical skill, as contrasted with some of the previously suggested methods of analysis, for example, that of W. von Koppenfels [Math. Ann. **105**, 694–706 (1931)]. In the second part of the paper the treatment is extended to the case where the line has an arbitrary initial distribution of voltage and current. *R. M. Foster.*

Rice, S. O. Filtered thermal noise—fluctuation of energy as a function of interval length. J. Acoust. Soc. Amer. **14**, 216–227 (1943). [MF 8194]

Let a source of random noise be connected to the input of a selective device (filter) with the amplitude characteristic $A(f)$, where f is the frequency, such that $\int_0^\infty A^2(f)df$ is finite. Let $I(t)$ be the instantaneous output. The quantity $E(t_1, T)=\int_{t_1}^{t_1+T}I^2(t)dt$ is proportional to the energy dissipated in the output between the moments t_1 and (t_1+T) and is a random variable if the starting time t_1 be regarded as chosen at random. Similarly, the energy difference for a time

separation S:

$$E(t_1, T)-E(t_1+S, T)=\Delta_S E=\int_{t_1}^{t_1+T} I^2(t)dt-\int_{t_1+S}^{t_1+S+T} I^2(t)dt$$

is a random variable. The mean value m_T of the random distribution $E(T)$ and the standard deviations σ_T and $\sigma_{T,S}$ of $E(T)$ and $\Delta E(T, S)$, respectively, are computed, and, in particular, the ratios σ_T/m_T and $\sigma_{T,T}/m_T$ for a narrow "ideal" band filter. The validity of the basis of the statistical computations is admittedly not rigorously established though made plausible in an appendix. The basis representation is $I_n(t)=\sum_{n=0}^{\infty} C_n \cos(\omega_n t-\varphi_n)$, where $\varphi_0, \varphi_1, \cdots, \varphi_N$ are random angles,

$$\omega_n=2\pi f_n, \quad f_n=n\cdot\Delta f, \quad C_n=\text{const}\cdot A(f_n)(2\Delta f)^{\frac{1}{2}},$$

and the basic method is to compute the statistical constants under the assumption of random phases φ_n and afterwards let $\Delta f\to 0$, $N\to\infty$. In this way, the standard deviations are obtained as double integrals over f involving the frequency characteristic $A(f)$. *H. G. Baerwald.*

Gross, B. On the principle of superposition in the theory of linear electric circuits. Univ. Nac. Tucumán. Revista A. 3, 121–123 (1942). (Portuguese) [MF 8147]

A convolutional expression of the charge in a mesh of an electrical network is proved by a new method.

I. Opatowski (Chicago, Ill.).

Higgins, Thomas James. New formulas for the inductance and reactance of square tubular conductors. J. Appl. Phys. 14, 185–187 (1943).

Lifshitz, E. M. On the theory of phase transitions of the second order. I. Changes of the elementary cell of a crystal in phase transitions of the second order. Acad. Sci. USSR. J. Phys. 6, 61–74 (1942). [MF 7414]

In a phase transition of the second order, occurring at a Curie point in the p,T-plane (without emission or absorption of latent heat), a continuous change in the arrangement of the atoms in an elementary crystal produces a discontinuous change in its symmetry. Let the original symmetry group (with higher symmetry) be $\mathfrak{G}_0$, the original density function ρ_0 and the original thermodynamic potential Φ_0, and let these become $\mathfrak{G}$, ρ and Φ after the transition. Then $\mathfrak{G}$ is a subgroup of $\mathfrak{G}_0$, and $\delta\rho=\rho-\rho_0$ is a linear combination $\sum c_i^{(n)}\varphi_i^{(n)}$ of functions $\varphi_i^{(n)}$ invariant under $\mathfrak{G}$ (but not under $\mathfrak{G}_0$) which are transformed among themselves in irreducible sets called "races," each race belonging to an irreducible representation of $\mathfrak{G}_0$. The functions $c_i^{(n)}$ of p and T must vanish at a Curie point and be infinitesimal in its neighborhood. The minimum condition for the potential requires that the quantity $\Phi-\Phi_0$, when expanded in a power series in the $c_i^{(n)}$, should start with quadratic terms, which by a suitable change of variables are reducible to a sum of squares $\sum A^{(n)}(c_i^{(n)})^2$. It further requires that all the functions $A^{(n)}(p,T)$ (one associated with each race) should be positive on one side of the Curie point where all $c_i^{(n)}$ vanish, but that one of these functions should change sign in passing through the Curie point. The nonvanishing $c_i^{(n)}$ then correspond to one race of $\phi_i^{(n)}$. To preserve stability at the Curie point, however, the third order terms must vanish, and the fourth order terms must be nonnegative. If the third order terms do not vanish identically, it is shown that the Curie point is isolated. Otherwise there is a curve of Curie points in the p,T-plane. The determination of all possible symmetry changes in phase transitions of the second order involves a study of the irreducible representations of the 230 space groups. These are associated with 14 different types of Bravais lattices, and the author, after discussing the representation of space groups in a general way, restricts himself to a study and enumeration of all the (75) possible changes of Bravais lattices which may occur at Curie points. Of particular importance are the changes in symmetry in which the number of symmetry elements is halved, and the size of the elementary cell is either left unchanged or is doubled. A figure illustrates the case for the simple cubic lattice. *J. S. Frame* (Meadville, Pa.).

Born, Max and Ledermann, Walter. Density of frequencies in lattice dynamics. Nature 151, 197–198 (1943). [MF 8129]

Ledermann notes that, if

$$A=\begin{bmatrix} C & H \\ H' & H_0 \end{bmatrix}, \quad B=\begin{bmatrix} C & K \\ K' & K_0 \end{bmatrix}$$

are two borderings with p rows and columns of the same real symmetric matrix C, the number P of latent roots of A in a given real interval differs from the number Q of latent roots of B in this interval by at most $2p$. If p is small compared with the order n of C, then P and Q are asymptotically equal. Application is made to the problem of crystal vibration, where n is the number of internal atoms and p the number of boundary atoms. The introduction of the cyclic boundary condition causes a change only in the bordering elements of the matrix and hence does not appreciably change the density of the latent roots.

Born quotes the above results in refutation of an attack by Raman and Nagendra Nath upon the foundations of lattice dynamics, in particular, upon the thermal theory of X-ray scattering. *C. C. MacDuffee* (New York, N. Y.).

Born, Max. The thermodynamics of crystal lattices. I. Discussion of the methods of calculation. Proc. Cambridge Philos. Soc. 39, 100–103 (1943). [MF 8112]

Following a review of the results obtained by the author and his collaborators in a series of papers on the stability of crystal lattices, a new program is outlined for the theoretical calculations of thermodynamic properties of crystals, including the temperature dependence of the elastic constants. The free energy per atom of a crystal can be expressed as $(\log\Delta)_m$, where Δ is the secular determinant for the vibrations belonging to a definite wave vector in the reciprocal space and the average is to be taken over all wave vectors. The elements of Δ will depend on the strain components, and the process of averaging for a strained crystal with correspondingly lowered symmetry leads to an extremely difficult analytical problem. However, numerical calculations have been performed for simple cases [see paper III of this series; cf. the following two reviews]. They show that for certain types of interaction forces no significant errors are committed if the averaging is performed on the elements of the determinant in place of the log of the determinant. In this way manageable formulae are obtained which can be expected to give a fairly accurate representation of the temperature dependence of the elastic constants.

L. W. Nordheim (Durham, N. C.).

Born, Max and Bradburn, Mary. The thermodynamics of crystal lattices. II. Calculation of certain lattice sums occurring in thermodynamics. Proc. Cambridge Philos. Soc. 39, 104–113 (1943). [MF 8113]

[Cf. the preceding and the following reviews.] The treatment of the thermodynamic properties of crystals involves

lattice sums of the type

$$S_n^{k_1k_2k_3}(\alpha)=\sum_l{}'\frac{l_1^{k_1}l_2^{k_2}l_3^{k_3}}{l^n}e^{-i(l\cdot\alpha)},$$

with

$$l=l_1^2+l_2^2+l_3^2 \quad\text{and}\quad l\cdot\alpha=l_1\alpha_1+l_2\alpha_2+l_3\alpha_3,$$

where α_1, α_2, α_3 are the phases of a wave. The summation is over all integer values for l_1, l_2, l_3 with exclusion of the (000) term. Methods are worked out for the transformation of such sums into rapidly converging series by means of theta functions and their transformations. Tables of these sums as functions of the phases are given for n from 8 to 16 and the lowest k indices. *L. W. Nordheim* (Durham, N. C.).

Bradburn, Mary. The thermodynamics of crystal lattices. III. The equation of state for a face-centered cubic lattice. Proc. Cambridge Philos. Soc. **39**, 113–127 (1943). [MF 8114]

The equation of state for a face centered cubic lattice is evaluated largely by numerical methods under the assumption of power laws for attractive (exponent m) and repulsive (exponent n) interatomic forces. The case $m=6$, $n=12$ is worked out in detail. The results depend essentially on the number of neighbors taken into account for the calculation of lattice sums in the static terms but show that it is sufficient to use nearest neighbors only in the thermal term. It is shown that the interchange of averaging described in paper I of the series [cf. the preceding two reviews] can be made without involving an error larger than that arising from the uncertainty about the atomic force law.
L. W. Nordheim (Durham, N. C.).

Martin, D. On the methods of extending Dirac's equation of the electron to general relativity. Proc. Edinburgh Math. Soc. (2) **7**, 39–50 (1942). [MF 7493]

The methods of extending Dirac's equation to general relativity used by Fock, Cartan, Ruse and Whittaker are reviewed and it is shown that they lead to the same equation.
A. H. Taub (Princeton, N. J.).

Jaeger, J. C. The energy loss by radiation of fast electrons in a Coulomb field. Proc. Cambridge Philos. Soc. **39**, 127–130 (1943). [MF 8115]

The energy loss by radiation of an electron in a Coulomb field is determined using the relativistic Coulomb wave functions. The method is an extension of that used in earlier papers by the author and Hulme [Proc. Roy. Soc. London. Ser. A. **148**, 708–728 (1935); **153**, 443–447 (1936)], and the mathematical details are omitted. Numerical values of the cross section per unit energy range for spontaneous transitions from a unit beam are compared with those obtained by the Born approximation.
M. C. Gray (New York, N. Y.).

Petiau, Gérard. Sur les équations d'ondes des corpuscules à spins entiers. C. R. Acad. Sci. Paris **214**, 610–612 (1942). [MF 7897]

From spinor equations of the Dirac type involving a completely symmetric spinor with $2n$ indices tensor equations involving a tensor with n indices are derived. The latter are those given by Fierz for a particle of spin n. The method for converting spin indices into tensor ones involves the use of the anti-symmetric matrix whose product with the sixteen linearly independent matrices obtained from the Dirac matrices and their products is a set of six anti-symmetric matrices and ten symmetric ones. *A. H. Taub.*

Soonawala, M. F. The structure of atomic nuclei. Indian J. Phys. **16**, 291–305 (1942). [MF 8234]

The possibility of atomic nuclei consisting of smaller nuclei of comparable masses is discussed. The constituent nuclei are assumed to be those of the rare gases, and they are supposed to be bound together by forces arising from the exchange of a heavy electron. The expected masses and charges of the exchange particle are calculated for some cases, and the nuclear spins of some of the rare gases are derived. The results are inconclusive and do not justify the very arbitrary assumptions on which the theory is based.
S. Kusaka (Princeton, N. J.).

Serber, R. and Dancoff, S. M. Strong coupling mesotron theory of nuclear forces. Phys. Rev. (2) **63**, 143–161 (1943). [MF 8135]

The work of Pauli and Dancoff [Phys. Rev. (2) **62**, 85–108 (1942); these Rev. **4**, 95] with the assumption of strong coupling between the meson field and the nuclear particle is extended to the case where two nuclear particles are present in order to obtain a theory of nuclear forces in this theory. Two types of meson fields, the charged scalar and the neutral pseudoscalar, are considered in detail, but the results are quite general. It is shown that, for large enough separation of the heavy particles, this theory gives nuclear forces of exactly the same form as those given by the perturbation treatment except for a numerical factor, $\frac{1}{4}$ for the charged scalar, and $\frac{1}{9}$ for the neutral, charged and symmetrical pseudoscalar meson fields. However, for small separations, the minimum of the potential energy of the system occurs when the isotopic spins or the spins of the heavy particles, depending on whether the meson field is the charged scalar or the neutral pseudoscalar, are "frozen" in parallel or anti-parallel directions, respectively. Also the interaction potentials have singularities as bad as those of the perturbation theory. Oscillations about the "frozen" position are investigated in order to estimate the distances at which the "frozen" system begins to "thaw." It is found that this distance is too large to account for the observed properties of the nuclear forces. *S. Kusaka.*

Schrödinger, E. Pentads, tetrads, and triads of meson-matrices. Proc. Roy. Irish Acad. Sect. A. **48**, 135–146 (1943). [MF 8128]

Schrödinger, E. Systematics of meson-matrices. Proc. Roy. Irish Acad. Sect. A. **49**, 29–42 (1943). [MF 8358]

In the first of these papers the equations describing a meson of spin one are written in five dimensional form. As has been shown by Duffin [Phys. Rev. (2) **54**, 1114 (1938)], who did not use the five dimensional notation, these equations lead to a set of matrices satisfying

$$(*)\qquad \begin{aligned} &\beta_l^3=\beta_l \\ &\beta_l\beta_m^2+\beta_m^2\beta_l=\beta_l &&l\neq m \\ &\beta_l\beta_m\beta_n+\beta_n\beta_m\beta_l=0 &&l\neq m,\ n\neq m \end{aligned}$$

It is shown that there are at most five ten dimensional matrices of rank six satisfying (*) and at most four five dimensional ones of rank two. In the second paper properties of the basis of the algebra generated by matrices satisfying (*) are studied for the five and ten dimensional representation. In the latter case the following quantities are constructed: 2 invariants, 2-four vectors, an antisymmetric tensor, and 2 symmetric tensors. *A. H. Taub.*

Lucas, René. Pression osmotique et diffusion. C. R. Acad. Sci. Paris **214**, 536–538 (1942). [MF 7900]

Christianovitch, S. A. On the motion of a gas-liquid mixture in porous media. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 277–282 (1941). (Russian. English summary) [MF 7712]

The author shows that in many instances the equations of Muskat and Meres for the motion of a gas-liquid mixture in a porous medium can be reduced to the usual equation for an incompressible fluid which follows the law of Darcy. *S. Bergman* (Providence, R. I.).

Darrow, Karl K. Memorial to the classical statistics. Bell System Tech. J. **22**, 108–135 (1943). [MF 8068]

A clear and forceful presentation of classical statistical mechanics according to Maxwell and according to Boltzmann. The emphasis is on the physical ideas, a modicum of combinatorial analysis and a minimum of formulas. After subjecting the developments to a critique, the author finds what he calls the rift in the lute in the classical theory; namely, if two different gases are allowed to interdiffuse, the entropy increases; yet if the two gas masses are formed of the same kind of gas, no such increase in entropy is envisaged, in spite of the fact that there are the same purely combinatorial grounds for an increase in the second case as in the former. This is on the assumption that the molecules retain their individuality and hence can be labeled. Thus the way is prepared for the introduction of quantum statistics, which is to form the subject of a subsequent expository article. *B. O. Koopman.*

Khintchine, A. Lois de distribution des fonctions sommatoires dans la mécanique statistique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 55–57 (1942). [MF 7598]

This is the fourth in a series of papers which appeared in the same C. R. and which are at present not available for review.

Vladimirskij, V. On the calculation of mean values of the product of two magnitudes corresponding to different moments in statistical mechanics. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **12**, 199–202 (1942). (Russian) [MF 7591]

Under the assumption of the Gibbs canonical distribution in phase an average of the type $\overline{A(t+\tau)B(\tau)}'$, where A and B are two functions in the phase space, is expressed in terms of the average of A, the average of B relative to the same canonical distribution and an average of A dependent on t and calculated with respect to a special distribution introduced for the purpose. Application is made to the case in which empirical equations of motion for the average values of the quantities involved are known. *I. Opatowski.*

Jaffé, George. A statistical theory of liquids. I. Phys. Rev. (2) **62**, 463–476 (1942). [MF 7472]

The author attempts a statistical theory of a two-phase system consisting of a liquid and its vapor. His aim is to avoid the great mathematical complications that arise for liquids in the rigorous partition method used by Mayer, Born and their co-workers, and also to refrain from the somewhat arbitrary though partly successful assumptions made in the cell-method used by Lennard-Jones, Eyring and others, which attribute too much of a crystal character to the liquid. The author gives a hybrid theory, based on a statistical theory of the kinetic and potential energy of the (point-) molecules, but also introducing an interface between the two phases, so that the potential energy of a molecule depends not only on the closely adjacent molecules but also on the distance from the surface. "This procedure constitutes in a certain way a return to very much older methods." The theory depends on a parameter σ, an intermolecular distance, which is treated as depending on the density and temperature of the medium. The method gives expressions for the vapor pressure and the surface tension. The existence of two phases and an interface is not simply assumed; having worked out the potential energy for such a system (under no external force) the conditions consistent with the existence of the system are determined. The dependence of σ on the density and temperature is determined by a second application of statistical principles to the immediate neighborhood of each molecule, somewhat on the lines of the treatment of electrolytes by Debye and Hückel. Finally, formulae are obtained for the characteristic coefficients of the liquid state, disregarding the influence of surface tension; these coefficients consist of the compressibility b, the coefficient of expansion a, the entropy and internal energy, the specific heat at constant volume c_v and the latent heat of vaporization L. These quantities can be numerically evaluated for molecules which interact in the manner discussed by Lennard-Jones, namely, $\varphi|r|=fr^{-\lambda}-gr^{-\mu}$; the four constants f, g, λ, μ and the equilibrium distance σ_0 may be determined from five experimental data and can then be used to predict further experimental data. This is done for ten liquids of widely different character, with satisfactory agreement (except in the case of water), having regard to the very simple molecular model considered. The agreement is regarded, however, only as indicating the adaptability, not the correctness, of the theory. A later paper is to extend the tests of the theory to the vapor pressure law and the surface tension. *S. Chapman* (London).

BIBLIOGRAPHICAL NOTE

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Volume 3, no. 2 [March 1943] contains reprints of mathematical papers [cf. these Rev. 3, 224]. Volume 2 and Volume 3, no. 1 are devoted to nonmathematical subjects.

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